The 2-gap case (where κ and κ' are two cardinals apart from λ and λ' respectively) the situation without $V=L$ is less decided and even GCH and λ' regular can not guarantee the existence of a (κ', λ') model (Silver). But again $V=L$ implies the existence of (κ', λ') model of T using the existence of a combinatorial creature named morass (well deserving its name). That there are morasses, in L is shown in Chapter 13, and in Chapter 14 morasses are used to get the 2-gap result in L. Similar arguments by sinking deeper into the morasses can give an n-gap two cardinal result in L. These results (due to Jensen, of course) set a record in their technical subtlety and it is the first time they are published anywhere. Devlin is doing an important service by publishing them.

The book now explores the implications of the existence of large cardinals on L. The most remarkable fact is that L is inconsistent with very large cardinals (due originally to D. Scott) and if we assume the existence of these cardinals $V=L$ is very badly violated. Again, one of the theorems, of Silver this time, is published here for the first time.

The book concludes by a study of relative constructibility and by showing that the class of sets constructible from a given set is similar in many respects to L and more so if we consider the sets constructible from a normal measure on a measurable cardinal. (In particular, we get a Souslin and Kurepa tree in such a universe.) The Herbacek-Vopenka Theorem, claiming that if there exists a strongly compact cardinal then the universe is not even constructible from a set, is proved.

The aims of the Springer-Verlag Lecture Notes in Mathematics states that "The timeliness of a manuscript is more important than its form, which may be unfinished or tentative." Devlin did not use this option: the standard of exposition in this book is high, and the presentation is very coherent and clear. Though there are places (like the definition of the projection) where more intuitive motivation is highly desirable, the book is an important source for any mathematician seriously interested in the subject.

REFERENCE

1. Gian-Carlo Rota, *Review of mathematical thought from ancient to modern times*, Bull. Amer. Math. Soc. **80** (1974), 805.

MENACHEM MAGIDOR

Analyse convexe et problèmes variationelles, by I. Ekeland and R. Teman, Dunod, Gauthier-Villars, Paris 1974, ix+340 pp.

This book gives a systematic exposition of the modern theories of the calculus of variations and optimal control. Of course the theory of convex sets and functions plays a very important role and the book begins with an elegant exposition of the theory of convex functions. A relatively new notion is that of the *polar* (or *conjugate*) *function* of a given (usually convex) function. The notion of a *dual problem* is a central theme of the book. There are many situations in which the primal (i.e., given) problem has no solutions but the dual does; this leads to a new notion of generalized solutions. The application of these ideas to the Dirichlet problem for minimal hypersurfaces in non-parametric form is particularly interesting. Other applications are given to several problems and to the theories of elasticity, mathematical economics, and optimal control. The developments seem rather complicated and detailed at times, but the book is interesting on the whole and should be useful to those working in a wide variety of fields. In view of the great amount of detail presented, we shall restrict ourselves to the discussion of the least complicated and/or the most typical examples in each chapter.

The book begins with an elegant treatment of the theory of convex sets and functions. The notions of supporting linear functions and the graphs of functions are introduced early and treated so as to get the greatest generality. The notion of two linear topological spaces "*in duality*" is introduced; a special case is a normed space V and its topological dual V^* in which the duality, denoted by $\langle u, u^*\rangle$, is given by $\langle u, u^*\rangle = u^*(u)$ ($u \in V$, $u^* \in V^*$). Given a pair of spaces in duality, the notion of a *polar* (or *conjugate*) *function* F^* of a given function F is defined by the formula

$$F^*(u^*) = \sup_{u \in V}\{\langle u, u^*\rangle - F(u)\}. \tag{1}$$

Convex functions and those which are upper envelopes of families of affine, continuous functions play important roles. In case $V = V^*$, both being finite dimensional Hilbert spaces with the usual duality, then F^* is the usual support function, if F is convex. All functions considered are defined over the whole space V (or V^*, etc.) and then are allowed to have the values $\pm\infty$; the set of points u of V where $F(u) < +\infty$ is called the *effective domain* of F and is denoted by dom F. The set $\Gamma(V)$ is the set of all functions, each of which is the upper envelope of some family of continuous, affine functions; the set $\Gamma_0(V)$ consists of $\Gamma(V)$ with the constants removed; it turns out that if $F \in \Gamma(V)$, then F is convex and lower-semicontinuous and if $F(u) = -\infty$ for some u, then $F(u) \equiv -\infty$. If $F \in \Gamma(V)$, then $F^{**} = F$ and, in all cases, $F^{***} = F^*$. The notions of *directional derivative, Gateaux differential, subdifferentiability*, etc., are defined and compared. F is *Gateaux differentiable* at u $(\in V) \Leftrightarrow \exists\, a\, u^* \in V^* \ni$

$$F'(v; u) = \lim_{\lambda \to 0^+} \frac{F(u + \lambda v) - F(u)}{\lambda} = \langle v, u^*\rangle \quad \forall v \in V.$$

Then u^* is called the *Gateaux differential* and is denoted by the symbol $F'(u)$. The notion of Gateaux differentiability is important in the calculus of variations. Suppose F *is Gateaux differentiable on a convex set* $\mathcal{A}$. *Then* F *is convex* $\Leftrightarrow F'$ *is a monotone map from* V *into* V^*.

In Chapter II, a number of existence theorems in the calculus of variations are proved by the so-called direct methods. For example, if F *is convex and lower-semicontinuous on a closed convex set* $\mathcal{C}$ *and either* $\mathcal{C}$ *is bounded or* F *is coercive on* $\mathcal{C}$ (i.e., $F(u) \to +\infty$ as $\|u\| \to \infty$ in $\mathcal{C}$), *then* F *takes on its minimum*; *if* F *is strictly convex the minimizing point is unique.*

Chapter III begins with the definition of the *dual problem*, denoted by P^*, corresponding to a given problem, denoted by P; the problem P is usually that of minimizing a function F and P^* is that of maximizing some function F^*. If we are given a problem P to minimize $F(u)$ for $u \in V$, we define P^* as follows: Let Y and Y^* be two spaces in duality and we choose $\Phi: V\times Y \to \bar{R}$ where $\Phi(u, 0)=F(u)$. We then let $\Phi^*: V^*\times Y^* \to \bar{R}$ be the polar function of Φ with respect to the duality between $V\times Y$ and $V^*\times Y^*$ (see (1) above), i.e.,

$$\begin{aligned}\Phi^*(u^*, p^*) &= \sup_{(u,p)\in V\times Y} \{\langle (u, p), (u^*, p^*)\rangle - \Phi(u, p)\} \\ &= \sup_{(u,p)\in V\times Y} \{\langle u, u^*\rangle_V + \langle p, p^*\rangle_Y - \Phi(u, p)\}.\end{aligned} \tag{2}$$

We then define the dual problem

$$P^*: \sup_{p^*\in P^*} \{-\Phi^*(0, p^*)\}. \tag{3}$$

From its definition, we see that P^* and $\sup\{-\Phi^*(0, p^*)\}$ depend on the choices of Y, Y^* and $\Phi(u, p)$. However it is shown in this chapter that there are many cases where the number above is independent of how Φ, Y, and Y^* are defined as long as $\Phi(u, 0)=F(u)$. A *Lagrangian* is introduced by the formula

$$-L(u, p^*) = \sup_{p\in Y} [\langle p^*, p\rangle - \Phi(u, p)]$$

and some of its uses are indicated; in particular it is shown that:

If $P: \inf_{u\in V} \sup_{p^*\in Y^*} L(u, p^*)$, *then* $P^*: \sup_{p^*\in Y^*} \inf_{u\in V} L(u, p^*)$, *for any* $\Phi \in \Gamma_0(V\times Y)$.

Some special results are proved for cases where $F(u)=J(u, \Lambda u)$ where Λ is a linear operator from V to Y; in all these cases

$$\inf(P) = \sup P^* \tag{4}$$

but sometimes only one of these problems has a solution. In (4), if P is the problem of finding $\inf F(u)$ for $u \in V$, then we define

$$\inf_{u\in V} P = \inf_{u\in V} F(u)$$

and $\bar{u}$ is said to be a *solution* of $P \Leftrightarrow F(\bar{u}) = \inf_{u\in V} F(u)$. Similarly if P^* is the problem of finding $\sup F^*$, then we define

$$\sup_{p^*\in Y^*} P^* = \sup_{p^*\in Y^*} F^*(p^*)$$

and $\bar{p}^*$ is said to be a solution of $P^* \Leftrightarrow F^*(\bar{p}^*) = \sup F^*(p^*)$. An important special subclass of these is one in which F is replaced by a function of the form $F(u)+G(\Lambda u)$; for such problems the dual problem P^* is to find $\sup_{p^*\in Y^*}\{-F^*(\Lambda^* p^*) - G^*(-p^*)\}$ where Λ^* is the transpose of Λ and is a linear operator from Y^* into V^*. The study of this class involves a rich formalism.

In Chapter IV, some special problems in the calculus of variations involving the use of Sobolev spaces are studied by reducing them to the abstract theorems in Chapter III; these are generalized Dirichlet problems. One problem was seen to be so reducible in two different ways. Finally, it is shown how to reduce a more general class of variational problems to those treated earlier in Chapter III using the duality idea.

In Chapter V, the duality set up of Chapter III is applied to the study of the Dirichlet problem for minimal hypersurfaces in nonparametric form (and similar problems). The primal problem P is that of minimizing $\int_\Omega \sqrt{1+|\text{grad } u|^2}\, dx$ among all u of the form $u=\phi+w_0^{1,1}(\Omega)$ where ϕ is a given function in $w^{1,1}(\Omega)$. The dual problem P^* is that of maximizing

$$\left[-\int_\Omega p^*(x) \text{grad } \phi(x)\, dx + \int_\Omega [1-|p^*(x)|^2]^{1/2}\, dx\right]$$

among all $p^* \ni p^* \in L^\infty(\Omega)^n$, $|p^*(x)| \leqq 1$ a.e. on Ω, $\operatorname{div} p^*=0$. It is shown using the general theories of Chapter III that *if Ω is sufficiently smooth and $\phi \in w^{1,1}(\Omega)$, then P and P^* are in duality,* $\inf P = \sup P^*$, *the problem P^* has a unique solution $\bar{p}^*$ (but P may not have any solution) which is analytic and locally bounded. In case P has a solution, $\exists$ an analytic function $\bar{u}$, uniquely determined up to an additive constant, which satisfies the differential equation for minimal surfaces and*

$$\text{grad } u(x) = -\frac{\bar{p}^*(x)}{(1-|\bar{p}^*(x)|^2)^{1/2}}, \qquad |\bar{p}^*(x)|<1, \quad x \in \Omega.$$

In Chapter VI, entitled "Duality by min-max", the authors consider problems P of the form: $\inf \Phi(u)$, where $\Phi(u)=\sup_{p\in Z} L(u,p)$. Thus the primal problem is

(5) $$P: \inf_{u\in V} \sup_{p\in Y} L(u,p).$$

We define the dual problem

(6) $$P^*: \sup_{p\in Y} \inf_{u\in V} L(u,p).$$

It is shown that if $F(u)=F_0(u)+F_1(u)$ where F_1 is convex, lsc, and "proper" on V, then there is a function L such that

(7) $$F(u) = \sup_{p\in\mathcal{B}} L(u,p).$$

Suppose $L(u,p)$ is defined on $\mathcal{A}\times\mathcal{B}$. We say that $(\bar{u},\bar{p})$ is a *saddle point* for L on $\mathcal{A}\times\mathcal{B} \Leftrightarrow L(\bar{u},p) \leqq L(\bar{u},\bar{p}) \leqq L(u,\bar{p})$ $\forall\, u\in\mathcal{A}$ and $p\in\mathcal{B}$. For any L, defined on a product set $\mathcal{A}\times\mathcal{B}$, we have

$$\sup_{p\in\mathcal{B}} \inf_{u\in\mathcal{A}} L(u,p) \leqq \inf_{u\in\mathcal{A}} \sup_{p\in\mathcal{B}} L(u,p).$$

Now suppose that $(\bar{u},\bar{p})$ is a saddle point for L on $\mathcal{A}\times\mathcal{B}$ and F is defined by (7). Then

$$F(\bar{u}) = \sup_p L(\bar{u},p) \leqq L(\bar{u},\bar{p}) \leqq L(u,\bar{p}) \leqq L(u,p) \leqq \sup_p L(u,p) = F(u)$$

so that $\bar{u}$ is a solution of the problem P. This shows the importance of this saddle-point approach. The following is a typical existence theorem for saddle points: *Suppose V and Z are reflexive Banach spaces, $\mathcal{A}\subset V$ and $\mathcal{B}\subset Z$ are convex, closed, and nonempty, $\forall\, u\in\mathcal{A}$, L is concave and upper semicontinuous in p, $\forall\, p\in\mathcal{B}$, L is convex and lower semicontinuous in u, and $\mathcal{A}$ and $\mathcal{B}$ are bounded. Then L possesses at least one saddle point $(\bar u, \bar p)$ on $\mathfrak{A}\times\mathcal{B}$ and*

$$L(\bar u, \bar p) = \operatorname*{Min}_{u\in\mathcal{A}} \operatorname*{Max}_{p\in\mathcal{B}} L(u, p) = \operatorname*{Max}_{p\in\mathcal{B}} \operatorname*{Min}_{u\in\mathcal{A}} L(u, p).$$

(Notice that the authors use Min instead of inf, etc.)

In Chapter VII, the authors are concerned with a number of special problems and with algorithms (due to Ugawa and Arrow-Hurwicz) which lead to solutions of some of the problems. One example is from the theory of numerical analysis and one from the theory of optimal control.

Chapter VIII begins with the definition and some discussion of normal integrands. If B is a Borel set in R^p and $f:\Omega\times B\to\bar R$, then f is called a *normal integrand*$\Leftrightarrow$for almost all x in Ω, $f(x,\cdot)$ is lsc on B and $\exists$ a Borel function $\tilde f:\Omega\times B\to\bar R \ni \tilde f(x,\cdot)=f(x,\cdot)$ for almost all $x\in\Omega$. The following obvious theorem is proved: *Suppose $f:\Omega\times R^p$ is a positive normal integrand from $\Omega\times R_p$ and $\{u_q\}$ is a sequence of measurable maps from Ω into R^p which converges a.e. to $\bar u$; then*

$$\int_\Omega f[x, \bar u(x)]\,dx \leqq \liminf \int_\Omega f[x, u_q(x)]\,dx. \tag{8}$$

Clearly in this theorem there is no requirement of convexity. However, *suppose we assume that*

$$\begin{gathered}\Phi: R^+\to\bar R,\quad \Phi(t)>0,\\ \Phi \text{ convex, with } t^{-1}\Phi(t)\to+\infty \quad\text{as}\quad t\to+\infty,\end{gathered} \tag{9}$$

$$f:\Omega\times R\times R^n,\qquad f(x, s, \xi)\geqq\Phi(|\xi|), \tag{10}$$

f is normal, and f is convex in ξ for almost all (x, s). Then

$$\int_\Omega f[x, \bar u(x), \bar p(x)]\,dx \leqq \liminf_{q\to\infty} \int_\Omega f[x, u_q(x), p_q(x)]\,dx \tag{11}$$

whenever $u_q(x)\to\bar u(x)$ a.e. and $p_q\to\bar p$ weakly in $L^1(\Omega)^n$.

The authors prove the following two theorems which do not require f to be convex in ξ: *Suppose that f is normal and satisfies* (10); *then f^{**} is normal and we also have*

$$\Phi(|\xi|)\leqq f^{**}(x, s; \xi). \tag{12}$$

(For the definitions of f^{**}, see Chapter 1.) *Suppose that f is normal and satisfies* (10). *Suppose $\{p_q\}$ converges weakly in $L^1_m(\Omega)$ to $\bar p$ and $\{u_q\}$ converges a.e. to $\bar u(x)$. Then*

$$\int_\Omega f^{**}[x, \bar u(x), \bar p(x)]\,dx \leqq \liminf_{q\to\infty} \int_\Omega f[x, u_q(x), p_q(x)]\,dx. \tag{13}$$

The authors prove the following interesting theorems: *Suppose f is normal and satisfies* (10) *and is convex in ξ for almost all (x, s) and G is a suitable* *linear operator from V into R^n. Then the problem*

$$(14) \qquad P: \inf_{u,p} \int_\Omega f[x, u(x), p(x)]\, dx, \qquad u = Gp, \quad p \in [L^1(\Omega)]^n,$$

has at least one solution. If f satisfies all the conditions above except the convexity in ξ, it is true that the problem

$$(15) \qquad PR \equiv P^{**}: \int_\Omega f^{**}[x, u(x), p(x)]\, dx$$

has at least one solution and

$$(16) \qquad \min(P^{**}) = \inf(P).$$

We can also say that if $(\bar{u}, \bar{p})$ with $\bar{u} = G\bar{p}$ is a solution of (PR), then there is a minimizing sequence (u_q, p_q), with $u_q = Gp_q$, for (P) such that $u_q \to \bar{u}$ a.e. and $p_q \to \bar{p}$ weakly in L_1 (actually more can be said about the convergence); *any such sequence contains a subsequence $(u_{q'}, p_{q'})$ such that $u_{q'} \to \bar{u}$ and $Gp_{q'} \to G\bar{p}$ in L_1.*

Chapter X extends and refines many of the results obtained in the preceding chapters. The fundamental problem of the calculus of variations

$$(P) \qquad \inf \int_\Omega f[x, u(x), \operatorname{grad} u(x)]\, dx, \qquad u - u_0 \in W_0^{1,\alpha}(\Omega),$$

receives special attention. The chapter concludes with a discussion of results involving the Euler equations and the problems (PR) (i.e., (P^{**})).

CHARLES B. MORREY, JR.

Characteristic classes by John W. Milnor and James D. Stasheff, Ann. of Math. Studies No. 76, Princeton Univ. Press, Princeton, New Jersey, 1974, vii+330 pp., $10.00

In 1957 there appeared notes by Stasheff of lectures on characteristic classes by Milnor at Princeton University. These notes are a clear concise presentation of the basic properties of vector bundles and their associated characteristic classes. Since their appearance they have become a standard text regularly used by graduate students and others interested in learning the subject.

The present, long-anticipated book is based on those notes. It follows the order of the notes but is considerably expanded with more detail and discussion. In addition, exercises have been added to almost each section, there are many useful references to the textbooks on algebraic topology that are available now, and there is an epilogue summarizing main developments in the subject since 1957. All of these strengthen the book and make it even more valuable as a text for a course as well as a book that can be read by students on their own. The material covered should be required for doctoral students in algebraic or differential topology and strongly recommended for those in differential geometry.