balanced, and a more widely appealing book, and it is a shame that the opportunity has been missed.

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Monotone matrix functions and analytic continuation, by W. F. Donoghue, Jr., Springer-Verlag, New York, Heidelberg, Berlin, 1974, 182 pp., $19.70

In a 1934 article Charles Loewner posed and solved the following problem: Characterize the class $P_n(a, b)$ of real-valued functions on the interval (a, b) that are *monotone matrix functions of order n*. This means that whenever A, B are n-by-n Hermitian matrices with spectrum in (a, b) and $A \geqq B$ (i.e. $A-B$ is positive definite), then $f(A) \geqq f(B)$. As usual, $f(A)$ is defined as the Hermitian matrix whose eigenvectors are the same as those of A and whose eigenvalues are gotten from those of A by applying f. Loewner showed that for $n \geqq 2$ such a function is automatically continuously differentiable and, regarded as a function from the linear space of n-by-n Hermitian matrices to itself, its derivative at $A = \mathrm{diag}(\lambda_1, \cdots, \lambda_n)$ sends the matrix (X_{jk}) to the matrix $([\lambda_j, \lambda_k]_f X_{jk})$, where

$$[x, y]_f = \begin{cases} \dfrac{f(x)-f(y)}{x-y} & \text{if } x \neq y, \\ f'(x) & \text{if } x = y. \end{cases}$$

So a necessary and sufficient condition for monotonicity of order n is the positive definiteness of the matrix $[\xi_j, \xi_k]_f$ for every choice of $\xi_1, \cdots, \xi_n \in (a, b)$. An equivalent condition is the positive definiteness of $[\xi_j, \eta_k]_f$ for every $a < \xi_1 < \eta_1 < \xi_2 < \cdots < \eta_n < b$; in fact Loewner starts with proving the necessity of this condition by a direct construction; it is from this that the differentiability of f follows. Having proved these results, Loewner proceeds to show the connections of matrices of the form $[\xi_j, \eta_k]_f$ with the Cauchy interpolation problem, i.e. the problem of interpolation on the complex plane by rational functions of sufficiently low degree. He then considers interpolation by monotone matrix functions, and finds among other things the following two striking results: (1) Monotonicity of order n is a local property (i.e. if it holds in two overlapping intervals then it holds in their union). (2) The intersection of all $P_n(a, b)$ $(n \geqq 1)$ is just the class $P(a, b)$ of functions analytic on (a, b) and analytically continuable to a *Pick function*, that is a holomorphic function mapping the entire upper halfplane into itself. As Loewner mentions in his article, he was led to this whole complex of questions by trying to find conceptual interpretations for certain inequalities true in $P(a, b)$.

In the fifties simpler ways were found to prove (2) directly from the positive definiteness of the kernel $[x, y]_f$. The Bendat-Sherman proof uses a transformation of the condition due to O. Dobsch and involving higher derivatives, then Bernstein's theorem on completely monotone functions to establish analyticity, then the Hamburger moment problem to get an integral representation from which the result can be read off. In the reviewer's proof one constructs the Hilbert space associated to the kernel $[x, y]_f$ and shows that f is equal to a diagonal matrix entry of the resolvent of a certain selfadjoint operator, from which again everything follows.

About the same time important new results were obtained by Loewner who looked at the problem again, now using the tools he introduced in his famous 1923 paper on the Bieberbach conjecture. The classes $P(a, b)$ form a pseudo-semigroup under composition of functions ("pseudo" because two functions can only be composed if the domain of one contains the range of the other). Just as in the theory of Lie transformation groups one can then define "infinitesimal transformations" and one-parameter semigroups, or a little more generally, paths emanating from the identity whose tangent at any point is an infinitesimal transformation. Loewner showed that the functions in $P(a, b)$ that can be approximated by functions lying on such paths are exactly those whose analytic continuation to the upper halfplane is univalent. Later he showed that for such functions f the kernel $\log[x, y]_f$ is *almost positive*; this means that for all $\xi_1, \cdots, \xi_n$ in (a, b) and for any complex numbers $z_1, \cdots, z_n$ *such that* $\sum z_j = 0$, we have $\sum \log[\xi_j, \xi_k]_f z_j \bar{z}_k \geqq 0$. The converse of this result was proved in 1967 by C. FitzGerald. R. A. Horn gave another proof avoiding semigroups, later he found a further criterion for a real function to have a univalent continuation. FitzGerald also returned to the subject in 1969 and proved a criterion for the univalent analytic continuation to have a starlike range.

W. F. Donoghue's book is a self-contained and unified account of these developments together with some introductory chapters on Pick functions and related matters. Just as in his papers, Donoghue writes very clearly and his presentation is highly polished in the sense that he gives great attention to finding the simplest, most elegant proofs even of small details. This book is also quite elementary. It uses a little Hilbert space theory and some basic complex variable theory; apart from these it proves practically everything it ever uses (with perhaps only the exception of the Hamburger moment problem).

To describe the contents in some detail: There are nineteen chapters, of an average length of less than ten pages each. The first contains elementary material on positive definite matrices and Newton interpolation, and a proof of Bernstein's theorem on completely monotone functions. The second contains a discussion of Pick functions, the Riesz-Herglotz theorem, the integral representation of Pick functions and of $P(a, b)$ derived from it, and a detailed discussion of the Gamma function based on the fact that $(\log \Gamma)'$ is a Pick function. In the third chapter is presented the old observation of G. Pick which probably started the whole subject: If φ is a Pick function and $z_1, \cdots, z_n$ are in the upper halfplane then $(\varphi(z_j)-\varphi(z_k))/(z_j-\bar{z}_k)$ is positive definite. (Loewner's positive definiteness condition is a limiting case of this.) An interesting result in the reverse direction due to A. Hindmarsh is also proved. The next chapter is a digression on Fatou's theorem whose interest in this context is that it makes more precise the relationship between a Pick function and the measure occurring in its integral representation. Chapter 5 contains a version of the proof of the spectral theorem of unbounded selfadjoint operators which is based on the fact that every diagonal entry of the resolvent is a Pick function.

It is after these preliminaries that the discussion of monotone matrix functions starts. In Chapter 6, which is only four pages long, there is a very simple and elegant proof of Loewner's observation that if $\xi_1<\eta_1<\xi_2<\cdots<\eta_n$ and the ξ_i are the eigenvalues of a given symmetric (or Hermitian) matrix, then there exists $c>0$ and a one-dimensional projection P such that the η_i are the eigenvalues of $A+cP$. In the next two chapters Loewner's characterization of monotone matrix functions of order n is presented; also some further equivalent conditions due to O. Dobsch are described. The proof of the sufficiency of these, however, depends on the result proved only much later that monotonicity of order n is a local property. Chapter 9, "Loewner's Theorem" contains the Bendat-Sherman proof that $[x, y]_f$ positive definite implies $f \in P(a, b)$. Chapter 11 contains reworked versions of the reviewer's proof of the same result and the similar proof of the Pick-Nevanlinna interpolation theorem due to B. Sz.-Nagy and the reviewer; the necessary preliminaries on the Hilbert space associated to a positive definite kernel are developed in Chapter 10 closely following the original presentation of N. Aronszajn. The next three chapters are a reworking of Loewner's results on interpolation by monotone matrix functions. They contain the original proof of Loewner's theorem and the proof that monotonicity of order n is a local property.

In Chapter 15 there is a general discussion of almost positive kernels. The class $Q(a, b)$ is then defined by the almost-positivity of $[x, y]_f$ and a simple connection between $P(a, b)$ and $Q(a, b)$ due to Horn is proved. The following three chapters are devoted to the Loewner-FitzGerald theorem on the univalency of the analytic continuation in $P(a, b)$. The part of the proof due to FitzGerald is considerably simplified, and both parts become much more transparent than in the original papers. A related result of Horn is also included. Finally, in Chapter 19 there is a result of Loewner characterizing Q_n as the class of infinitesimal transformations for P_n and a theorem of FitzGerald about when the range of a univalent function in $P(a, b)$ is starlike.

So the book contains practically everything that is known about the subject, collected, unified and simplified. It also has the beginner in mind, and makes various digressions to acquaint him with related topics. In this direction, however, I feel that more could have been done. It is only a slight overstatement to say that Donoghue either discusses a related topic thoroughly, with complete proofs, or he does not say anything about it at all. This is perhaps an exaggerated concern for formal perfection which is not without its disadvantages. Of course I am not advocating a mathematical style full of vague allusions, but I think that some brief descriptions of related subjects and methods with good references would have been helpful and could have been put in larger numbers into the section of "Notes and comment" which is at the end of the book anyway.

For example, monotone matrix functions of several variables and convex matrix functions are not mentioned, although at least the main references for the latter are in the bibliography. More importantly, the Hamburger moment problem, which is used in Chapter 9, is not discussed, there are only references to Shohat-Tamarkin and Widder. Since Chapter 5 contains a proof of the spectral theorem and Chapter 10 the theory of reproducing kernels, it would have taken only an extra paragraph to give a complete solution of the moment problem, one that would be very much in the spirit of the book. This would also illuminate further the comments after Chapter 13 on the limit-point and limit-circle case. In fact it could have been pointed out that these questions are equivalent with certain uniqueness questions about selfadjoint extensions or dilations of symmetric operators. Some of these connections become clear in the Hilbert space theory textbook of Akhiezer and Glazman, which together with the more up-to-date book of Akhiezer on the moment problem could well be included in the list of references.

These objections are of course not overly serious. The reader trying to get into the subject in depth will notice the connections with other topics anyway, and will in the meantime be greatly helped by Donoghue's book. It is likely that there will be many such readers since the subject is very attractive and also seems to be far from being closed. For example, the author mentions the open question whether membership in Q_n is a local property; it would also be interesting to find a simpler proof that membership in P_n is local. Another small question is whether by some method one could discuss monotonicity of arbitrarily high order directly, without passing through the case of fixed order n. There are also some very recent not yet published results of M. Rosenblum and J. Rovnyak on the connections of Loewner's theorem with the Hilbert transform; there may be some further questions worth exploring in the same direction.

All in all, the subject is certainly an interesting one, and Donoghue's book is a beautifully written, excellent account of it. It is to be highly recommended to expert and beginner alike.

As for mistakes, there do not seem to be many. The consistent misspelling of Hans Bremermann's name is probably nothing but the final proof that Springer has become a naturalized American publisher.

ADAM KORÁNYI

Homology in group theory, by Urs Stammbach, Lecture Notes in Mathematics, Volume 359, Springer-Verlag, New York, 1973, vii+183 pp., $7.00

Cohomology theory is still viewed by many group theorists with a mixture of suspicion and indifference. There are, of course, good reasons for this. Homology theory began to invade group theory just before the Second World War, but it was not until the early fifties that the theory had been satisfactorily rebuilt as a completely algebraic tool. Group theorists then discovered that they had actually been practising homological algebra for years without knowing it: in Schur's theory of covering groups; in extension theory as developed by Schreier and Baer; and even in Burnside's transfer map. But homological algebra did not seem to be more than a satisfactory setting for work already done.

This situation has begun to change in the last fifteen years. The reasons perhaps can be grouped under three headings. First of all, a number of distinctly nontrivial new results have been established by homological methods. Samples: the Stallings-Swan theorem that a group G is free precisely if every extension by G with abelian kernel is split; the result of Gaschütz that every finite p-group has an outer automorphism of order p; the very recent theorem of Bieri that a finitely presentable group, of cohomological dimension 2 and with a nontrivial centre, has a free commutator group.

Secondly, the outlook engendered by homological algebra can surely be held responsible for the spectacular development of integral representation theory, due to Swan and many others, as well as to important progress in modular representation theory such as Green's theory of sources and vertices.

Finally, the homological language has shown itself to be the natural one in which to express a good deal of the post-war work on generalised nilpotent groups.

Stammbach's book is, above all, a contribution under this third heading. A good two-thirds of it deals with centrality properties of groups: the lower central series (Chapter 4), central extensions (Chapter 5) and localization of nilpotent groups (Chapter 6). Probably a more realistic (albeit more cumbersome!) title for the book would have been *Homological methods in the study of nilpotency properties of groups.*

The treatment of these topics is smooth, coherent and often elegant. It should help to convince group theorists of the efficacy of homological