A COUNTEREXAMPLE IN SHAPE THEORY[1]

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In the process of trying to solve a certain problem in Harmonic analysis (cf. [5]) the author encountered the following shape theory question: If f: $X \longrightarrow Y$ is a surjective map of compact spaces and $f^{-1}(y)$ has trivial shape for each $y \in Y$, then is f necessarily a shape equivalence? We were able to prove that the answer is yes provided Y is finite dimensional, but this was of no use in the problem we were considering.

We discussed this question with R. D. Anderson, who pointed out that the problem has been known to specialists in shape theory (e.g., Anderson, Borsuk, Mardešić) for some time. Anderson has a proof (as yet unpublished) in the case where $\{y: f^{-1}(y)$ is nondegenerate$\}$ is finite dimensional. In case X and Y are both finite dimensional the result is essentially Theorem 11 of Sher's paper [4].

The purpose of this paper is to present a counterexample for the general question. Specifically, if Q is the Hilbert cube, we will construct a compact metric space X and a surjective map $f: X \longrightarrow Q$ such that $f^{-1}(q)$ has trivial shape for all $q \in Q$, but X does not have trivial shape. Since Q has trivial shape, this means that f is not a shape equivalence.

Our example depends heavily on a K-theory result of Adams. In fact, given Adams' result, ours is little more than an observation.

1. **Shape.** Borsuk [2] invented the concept of shape as a substitute for homotopy type when one is dealing with spaces which are not locally nice (also see [3]). We will not define the notion of shape equivalence here. For our purposes it is enough to know the following: If $f: X \longrightarrow Y$ is a continuous map between compact Hausdorff spaces, then f is a shape equivalence if and only if $f^*: [Y, M] \longrightarrow [X, M]$ is bijective whenever M is an ANR. Thus, if H is any contravariant functor from compact spaces to sets which has a

classifying space which is an ANR, then $f^*: H(Y) \longrightarrow H(X)$ is bijective if f: $X \longrightarrow Y$ is a shape equivalence. Since Čech cohomology and K-theory are such functors, a map which is a shape equivalence induces isomorphisms of Čech cohomology and of K-theory.

A space X has trivial shape if the injection of a point into X is a shape equivalence. A fact we shall need which is easy to prove is the following: If $X = \varprojlim X_\alpha$ is an inverse limit of compact spaces and each bonding map $X_\beta \longrightarrow X_\alpha$ is null homotopic, then X has trivial shape.

2. **The example.** According to Theorem 1.7 of Adams' paper [1], there exists a compact space F ($F = S^{2q-1} \cup_g I^{2q}$ for large q and $g: S^{2q-1} \longrightarrow S^{2q-1}$ a map of degree p for p an odd prime) and a map $h: S^r F \longrightarrow F$ such that $h^*: \widetilde{K}(F) \longrightarrow \widetilde{K}(S^r F)$ is an isomorphism. Here $S^r F$ is the rth suspension of F, $r = 2(p-1)$, $\widetilde{K}$ is reduced complex K-theory, and $\widetilde{K}(F) = Z_p$ is nontrivial.

By taking iterated rth suspension of the map h we obtain an inverse limit system

$$\ldots \longrightarrow S^{nr}F \xrightarrow{S^{(n-1)r}h} S^{(n-1)r}F \longrightarrow \ldots \longrightarrow S^{2r}F \xrightarrow{S^r h} S^r F \xrightarrow{h} F$$

which has a limit $X = \varprojlim S^{nr}F$ with nontrivial K-theory ($\widetilde{K}(X) = Z_p$) and, hence, nontrivial shape. However,

THEOREM. *There is a surjective map $f: X \longrightarrow Q$ with $f^{-1}(q)$ having trivial shape for each $q \in Q$. Here Q is the Hilbert cube, which obviously has trivial shape.*

PROOF. The mth suspension $S^m F$ of F can be regarded as the space obtained from $I^m \times F$ by identifying $\{t\} \times F$ to a point for each $t \in \partial I^m$. Hence, there is a surjection $f_m: S^m F \longrightarrow I^m$ with $f_m^{-1}(t)$ a point if $t \in \partial I^m$ and $f^{-1}(t) \simeq F$ if $t \in I^m \setminus \partial I^m$.

Consider the commutative diagram

$$\begin{array}{ccc} S^r F & \xrightarrow{\ h\ } & F \\ \downarrow{\scriptstyle f_r} & & \downarrow{\scriptstyle f_0} \\ I^r & \xrightarrow{\ \pi_1\ } & \{pt\} \end{array}$$

On passing to the cartesian product with I^r we obtain a commutative diagram

$$I^r \times S^r F \xrightarrow{\quad 1 \times h \quad} I^r \times F$$

$$\downarrow 1 \times f_r \qquad\qquad \downarrow 1 \times f_0$$

$$I^r \times I^r \xrightarrow{\quad 1 \times \pi_1 \quad} I^r$$

If we collapse fibers of $1 \times f_r$ and $1 \times f_0$ over points of $(\partial I^r) \times I^r$ and ∂I^r, respectively, we obtain a commutative diagram

$$S^{2r}F \xrightarrow{\quad S^r h \quad} S^r F$$

$$\downarrow f_{2r} \qquad\qquad \downarrow f_r$$

$$I^{2r} \xrightarrow{\quad \pi_2 \quad} I^r$$

where $\pi_2 = 1 \times \pi_1$. Obviously, we can iterate this procedure to obtain a commutative system

$$\cdots \longrightarrow S^{nr}F \xrightarrow{\ h_n\ } S^{(n-1)r}F \longrightarrow \cdots \xrightarrow{\ h\ } F$$

$$\downarrow f_{nr} \qquad\qquad \downarrow f_{(n-1)r} \qquad\qquad \downarrow$$

$$\cdots \longrightarrow I^{nr} \xrightarrow{\ \pi_n\ } I^{(n-1)r} \longrightarrow \cdots \xrightarrow{\ \pi_1\ } (pt)$$

where $h_n = S^{(n-1)r}h$. The vertical maps are all surjective and their limit is a surjective map $f\colon X \longrightarrow Q$ where $X = \varprojlim_n S^{nr}F$ and $Q = \varprojlim_n I^{nr}$ is the Hilbert cube.

If $t = \{t_i\} \in Q$ $(t_i \in I^{ir}, \pi_i t_i = t_{i-1})$ then $f^{-1}(t) = \varprojlim_n f_{nr}^{-1}(t_n)$ where the nth bonding map is the map $k_n\colon f_{nr}^{-1}(t_n) \longrightarrow f_{(n-1)r}^{-1}(t_{n-1})$ given by restricting h_n to $f_{nr}^{-1}(t_n)$.

Now each k_n is null homotopic. In fact, if $t_n \in \partial I^{nr}$ then $f_{nr}^{-1}(t_n)$ is a point. If $t_n \notin \partial I^{nr}$ then

$$h_n|_{f_{nr}^{-1}\pi_n^{-1}(t_{n-1})} \colon f_{nr}^{-1}\pi_n^{-1}(t_{n-1}) \longrightarrow f_{(n-1)r}^{-1}(t_{n-1})$$

is a copy of the map $h\colon S^r F \longrightarrow F$ and k_n is a copy of h restricted to $f_1^{-1}(s)$ for some $s \in I^r$. The restriction of h to $f_1^{-1}(\gamma)$ for γ an arc from s to ∂I^r yields a homotopy joining $h_{|f_1^{-1}(s)}$ to a constant map and, hence a homotopy from k_n to a constant map.

Since $f^{-1}(t)$ is an inverse limit with null homotopic bonding maps, it

follows that $f^{-1}(t)$ has trivial shape for each $t \in Q$.

Note that, in this example, each $f_{nr}^{-1}(t_n)$ is either a point or a copy of the finite dimensional space F. Hence, the fibers $f^{-1}(t)$ are of bounded dimension. In other words, the problem derives from the infinite dimensionality of the image space Q and has nothing to do with the dimensions of the fibers.

ADDED IN PROOF. R. D. Anderson, R. Edwards and J. Keesling (in chronological order) have each written to us pointing out that our example can be used to construct a cell-like map of the Hilbert cube onto a space which is not an absolute retract.

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