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WHEN IS A MANIFOLD A LEAF OF SOME FOLIATION?

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Given a connected smooth open manifold L, does there exist a compact manifold M and a C^r codimension q foliation of M with a leaf diffeomorphic to L? Here $1 \leqslant r \leqslant \infty$. Most of our results are for $q = 1$, but note that if the answer is yes for q then it is yes for any $q' > q$. Theorem 1 gives four conditions on L any one of which is sufficient, and the Corollary provides interesting examples where L is a surface. We have found no necessary condition in general, but Theorem 2 gives a strong necessary condition on the ends of L in order that L be a codimension one leaf each of whose ends has only one asymptote. Details and proofs will appear elsewhere.

THEOREM 1. *L is diffeomorphic to a leaf of a C^r codimension q foliation of some compact manifold if any one of the following conditions is satisfied* ($q = 1$ *except possibly in condition* 1.4).

1.1. *L is diffeomorphic to the interior of a compact manifold-with-boundary* ($r = \infty$ *and L will be a proper leaf*).

1.2. $L = L_1 \# L_2$ *where L_1 and L_2 are proper leaves of C^r codimension one foliations of compact orientable manifolds.*

1.3. $L = L_1 - X$ *where L_1 is a leaf of a C^r codimension one foliation of a compact manifold with a closed transversal which intersects L_1 in X.*

1.4. *L is a regular covering space of a compact manifold with covering group which has a C^r action on a connected compact q-manifold with a free orbit.* (*If the orbit is discrete, the leaf L will be proper.*)

Recall (see e.g. [2]) that an *end* ϵ of a connected manifold is determined by a sequence $U_1 \supset U_2 \supset \ldots$ of unbounded components of the complements of compact sets such that $\bigcap_{i=1}^{\infty} \overline{U}_i = \varnothing$. Another such sequence $V_1 \supset V_2 \supset \ldots$ determines the same end if every U_i contains some V_j. Each U_i is called a *neighborhood* of ϵ. Define ϵ to be *boundable* if it has a closed neighborhood of the form $B \times [0, \infty)$ where B is a connected compact manifold.

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COROLLARY. *Every orientable 2-manifold with a finite number of ends is a proper leaf of a C^r foliation of a compact 3-manifold, where $r = 1$ or $r = \infty$ depending on whether the number of nonboundable ends is odd or even, respectively.*

If ϵ is an end of a leaf L of a foliation of a manifold M, define the *asymptote set* A_ϵ of ϵ to be $\bigcap_{i=1}^{\infty} \mathrm{Cl}(U_i)$, where ϵ is determined by neighborhoods $U_1 \supset U_2 \supset \ldots$ in L and $\mathrm{Cl}(U_i)$ denotes the closure of U_i in M. Then A_ϵ is a well-defined closed union of leaves and is connected if M is compact. Define a leaf L to be *nice* if A_ϵ is a single leaf for every end ϵ of L. Note that a nice leaf is proper and that A_ϵ is compact if M is compact. Finally, say that an end ϵ of a manifold L is an *infinite repetition* if some closed neighborhood in L of ϵ is of the form $W \cup_f W \cup_f \ldots$ where W is a connected compact manifold-with-boundary, Bd W has two components $\mathrm{Bd}_- W$ and $\mathrm{Bd}_+ W$, and $f\colon \mathrm{Bd}_+ W \longrightarrow \mathrm{Bd}_- W$ is a diffeomorphism.

THEOREM 2. *If L is a nice leaf of a C^1 codimension one foliation of a compact manifold then L has only a finite number of ends and each one is an infinite repetition.*

The proof uses the following two theorems, of which the first is a generalization of Reeb's first stability theorem in [3] and the second is proved using the framed surgery method of [1].

THEOREM 3. *Let M be a (not necessarily compact) manifold-with-(possibly empty) boundary with a codimension q foliation transverse to* Bd M. *Let A be a compact leaf and let D be a q-disk transverse to the foliation and cutting A in exactly one point x_0. Suppose there exists a point x in D such that each element of the holonomy group of A has a representative local diffeomorphism of D whose domain contains x and which leaves x fixed. If x is sufficiently closed to x_0 then the leaf through x is diffeomorphic to A.*

THEOREM 4. *If $h\colon \Pi_1 A \longrightarrow \mathbf{Z}$ is a surjection, where A is a connected compact manifold, then there exists a smooth map $g\colon A \longrightarrow S^1$ such that $h = g_*\colon \Pi_1 A \longrightarrow \Pi_1 S^1 = \mathbf{Z}$ and for some regular value v in S^1, the manifold $g^{-1}(v)$ is connected and does not separate A.*

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