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THE RANGE OF A VECTOR MEASURE

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Let X be a real quasi-complete locally convex topological vector space. Let $K \subset X$ be a weakly compact convex and symmetric set such that $0 \in K$.

Let T be an abstract space and S be a σ-algebra of subsets of T. A vector measure is a σ-additive mapping m: $S \longrightarrow X$.

We are concerned with the question whether there exists a vector measure m: $S \longrightarrow X$ such that K coincides with the closed convex hull of the range of m, i.e. $K = \overline{\text{co}}\ m(S) = \overline{\text{co}}\{m(E): E \in S\}$. The case $X = R^n$ was surveyed in [1].

THEOREM 1. *If T is a space, S a σ-algebra of subsets of T and m: $S \longrightarrow X$ a vector measure, then there exists a space T_1, a σ-algebra S_1 of subsets of T_1 and a vector measure m_1: $S_1 \longrightarrow X$ such that*

$$\overline{\text{co}}\ m(S) = \overline{\text{co}}\ m_1(S_1) = \left\{\int_{T_1} f\, dm_1 : 0 \leqslant f \leqslant 1, f \text{ is } S_1\text{-measurable}\right\}$$

$$= \{m_1(E): E \in S_1\} = m_1(S_1).$$

It is worth mentioning that the equality $\overline{\text{co}}\ m(S) = \{\int f\, dm: 0 \leqslant f \leqslant 1\}$ does not hold, in general [3].

LEMMA. *If $K = \overline{\text{co}}\ m(S)$ and $y \in K$, then there exists a vector measure m_1: $S \longrightarrow X$ such that $K - y = \overline{\text{co}}\ m_1(S)$.*

In view of Theorem 1, the proof of this Lemma is not different from one given by Halmos in the case $X = R^n$ (see [1]). The Lemma permits us to restrict our attention to sets having 0 for the center of symmetry.

Assume that 0 is the center of symmetry of K. For any element $x' \in X'$, the continuous dual of X, let $\|x'\|_K = \sup\{|\langle x', x\rangle|: x \in K\}$.

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A function $\phi: X' \longrightarrow R$ is termed negative-definite if, for any collection $x'_1, x'_2, \ldots, x'_k$ of elements in X' and real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ such that $\Sigma_{j=1}^k \alpha_j = 0$, the inequality

$$\sum_{j=1}^{k} \sum_{l=1}^{k} \alpha_j \alpha_l \phi(x'_j - x'_l) \leqslant 0$$

holds.

The main result is expressed in

THEOREM 2. *Let* 0 *be the center of symmetry of* K. *There exists a vector measure* $m: S \longrightarrow X$ *such that* $K = \overline{\text{co}}\, m(S)$ *if and only if the function* $x' \longrightarrow \|x'\|_K, x' \in X'$, *is negative-definite.*

According to [2], the proof of Theorem 2 will be accomplished if we show that a set is a closed convex hull of the range of a vector measure if and only if it is a zonoform. This is the content of the last theorem.

We interpret X' as a set of real-valued functions defined on X. Denote by $h(X)$ the linear lattice of functions generated by X'. A nonnegative linear functional on $h(X)$ is termed a conical measure on X. The set of all conical measures on X is denoted by $M^+(X)$. There is a natural (partial) order in $M^+(X)$, viz. for $u, v \in M^+(X)$ we write $u \leqslant v$ if and only if $u(z') \leqslant v(z')$, for every $z' \in h(X)$, $z' \geqslant 0$.

Given $u \in M^+(X)$ and $x \in X$, we write $x = r(u)$ and call x the resultant of u if $\langle x', x \rangle = u(x')$, for every $x' \in X'$.

Let $m: S \longrightarrow X$ be a vector measure, where S is a σ-algebra of subsets of a set T. Denote by $ca(S)$ the linear lattice of all real-valued σ-additive measures on S. Let $\Phi_m: h(X) \longrightarrow ca(S)$ be the linear lattice homomorphism such that $\Phi_m(x') = x' \circ m$, for every $x' \in X'$. For every $z' \in h(X)$, let $u(z') = \Phi_m(z')\,(T)$. Then it can easily be shown that u is a conical measure. Denote it by $u = \Delta(m)$.

THEOREM 3. *If* $m: S \longrightarrow X$ *is a vector measure, then* $u = \Delta(m)$ *is a conical measure such that the resultant* $r(v)$ *exists for every* $v \in M^+(X)$ *with* $v \leqslant u$ *and*

$$\overline{\text{co}}\, m(S) = \{r(v) \colon v \in M^+(X), v \leqslant u\}.$$

For every conical measure u *such that* $r(v)$ *exists for each* $v \in M^+(X)$ *with* $v \leqslant u$ *there exists a space* T, *a* σ*-algebra* S *of subsets of* T *and a vector measure* $m: S \longrightarrow X$ *such that* $u = \Delta(m)$.

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