BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 81, Number 3, May 1975

SOME SINGULAR PERTURBATION PROBLEMS

BY F. A. HOWES[1]

Communicated by Hans Weinberger, December 27, 1974

1. Introduction. The singularly perturbed boundary value problem

$$\epsilon y'' = f(t, y, y', \epsilon), \quad 0 < t < 1, \tag{1.1}$$

$$y(0, \epsilon) = A, \quad y(1, \epsilon) = B, \tag{1.2}$$

for ϵ a small positive parameter, has been studied extensively under various linearity restrictions. See, for example, [3] and [4], and the references therein. However, two principal assumptions have been that the corresponding reduced problem

$$\begin{aligned} &0 = f(t, u, u', 0), \quad 0 < t < 1, \\ &u(1) = B, \end{aligned} \tag{1.3}$$

has a solution $u = u(t)$ of class $C^{(2)}[0, 1]$ and that in a suitable tube around u, $f_{y'} = \partial f/\partial y' \leqslant - k$, for some positive constant k. This latter assumption excludes the occurrence of turning points and makes the function u a stable root of (1.3).

Under additional assumptions, by means of several asymptotic methods, the existence of a solution $y = y(t, \epsilon)$ of (1.1), (1.2), for each ϵ sufficiently small, can be deduced and this solution can be shown to satisfy an estimate of the form

$$y(t, \epsilon) = u(t) + O(|A - u(0)| \exp[-kt\epsilon^{-1}]) + O(\epsilon), \quad 0 \leqslant t \leqslant 1.$$

Here O denotes the standard Landau order symbol. The exponential term $v(t, \epsilon) = \exp[-kt\epsilon^{-1}]$ is a boundary layer function, in that $v(0, \epsilon) = 1$ and $v(t, \epsilon) \longrightarrow 0$ as $\epsilon \longrightarrow 0^+$ for $t > 0$.

2. Statement of the problem and main result. Consider the more general boundary value problem

$$a(t, \epsilon) y'' = f(t, y, y', \epsilon), \quad 0 < t < 1, \tag{2.1}$$

AMS (MOS) subject classifications (1970). Primary 34E15; Secondary 34B15.

[1] Supported by the National Science Foundation under Grant no. NSF-GP-37069X.

(2.2) $$y(0, \epsilon) = A, \quad y(1, \epsilon) = B,$$

and the corresponding reduced problem

(2.3) $$a(t, 0)u'' = f(t, u, u', 0), \quad 0 < t < 1,$$

(2.4) $$u(1) = B,$$

where $a(t, \epsilon) = a(t, 0) + \tilde{a}(t, \epsilon)$, $\tilde{a}(t, \epsilon) > 0$ and $\tilde{a}(t, \epsilon) = O(\epsilon)$, for $(t, \epsilon) \in [0, 1] \times (0, \epsilon_1]$, $\epsilon_1 > 0$.

THEOREM. *Assume* (1) *the problem* (2.3), (2.4) *has a solution* $u = u(t)$ *of class* $C^{(2)}(0, 1) \cap C[0, 1]$;

(2) *the functions* $f, f_t, f_y, f_{y'}$ *are continuous in* R: $0 \leq t \leq 1$, $|y - u(t)| \leq d$, $|y'| < \infty$, $0 \leq \epsilon \leq \epsilon_1 (d, \epsilon_1 > 0)$;

(3) *there is a function* $b = b(t, \epsilon) > 0$, *for* $(t, \epsilon) \in [0, 1] \times (0, \epsilon_1]$, *such that* $f_{y'} \leq - b(t, \epsilon)$ *in* R;

(4) *there is a constant* $l > 0$ *such that* $f_y(t, y, u'(t), \epsilon) \geq l$ *for* $t \in (0, 1)$, $|y - u(t)| \leq d$ *and* $0 < \epsilon \leq \epsilon_1$;

(5) $\Gamma(a(t, \epsilon), b(t, \epsilon), \rho) = b^2a^{-1}(\rho - \rho^2) + a(ba^{-1})'\rho + l \geq 0$, *for some constant* $\rho > 0$ *and* $(t, \epsilon) \in (0, 1) \times (0, \epsilon_1]$;

(6) $|f(t, y, y', \epsilon)| \leq \phi(|y'|)$, *for* $t \in [0, 1]$, $|y| \leq M$, $|y'| < \infty$ *and* $0 < \epsilon \leq \epsilon_1$, *with* ϕ *positive, continuous and satisfying* $\int^{\infty} s\phi^{-1}(s)\, ds = \infty$;

(7) $f(t, u(t), u'(t), \epsilon) = f(t, u(t), u'(t), 0) + \tilde{f}(t, \epsilon)$, *for* $(t, \epsilon) \in (0, 1) \times (0, \epsilon_1]$;

(8) *there is a function* $\gamma = \gamma(t, \epsilon)$ *such that* $\gamma' \leq 0$ *and*

(i) $a(t, \epsilon)\gamma'' + b(t, \epsilon)\gamma' - l\gamma \leq \tilde{a}(t, \epsilon)u''(t) - \tilde{f}(t, \epsilon)$, *for* $(t, \epsilon) \in (0, 1) \times (0, \epsilon_1]$;

(ii) $\gamma > 0$ *and* $\gamma = O(\eta)$, $\eta = \eta(\epsilon) \to 0$ *as* $\epsilon \to 0^+$, *for* $(t, \epsilon) \in [0, 1] \times (0, \epsilon_1]$.

Then for each ϵ, $0 < \epsilon \leq \epsilon_1$, *there exists a solution* $y = y(t, \epsilon)$ *of* (2.1), (2.2). *In addition*,

$$y(t, \epsilon) = u(t) + O\left(|A - u(0)| \exp\left[- \rho \int_0^t (ba^{-1})(s, \epsilon) ds\right]\right) + O(\eta),$$
$$0 \leq t \leq 1.$$

The Theorem is proved by constructing Nagumo-type lower and upper solutions α, β, respectively. See, for example, [2]. As an illustration, if $u(0) \geq A$, the functions

$$\alpha(t, \epsilon) = u(t) - (u(0) - A)\exp\left[- \rho \int_0^t (ba^{-1})(s, \epsilon) ds\right] - \gamma(t, \epsilon),$$

$$\beta(t, \epsilon) = u(t) + \gamma(t, \epsilon)$$

satisfy the required inequalities.

3. **Discussion.** The Theorem includes a result like that mentioned in the Introduction, in that for $a(t, \epsilon) \equiv \epsilon$ and $b(t, \epsilon) \equiv k > 0$,

$$\Gamma(a(t, \epsilon), b(t, \epsilon), \rho = 1) \equiv l > 0.$$

Further, the assumptions that u is of class $C^{(2)}[0, 1]$ and $\tilde{f} = O(\epsilon)$ lead to the choice of $\gamma(t, \epsilon) \equiv \epsilon\sigma$, for σ a sufficiently large positive constant.

A more interesting situation occurs when $a(t, \epsilon) = (t + \epsilon)^2$ and $b(t, \epsilon) = k(t + \epsilon)$, $k > 0$. Then $\Gamma(a(t, \epsilon), b(t, \epsilon), \rho) \equiv 0$, provided $\rho = -rk^{-1}$, where r is the negative root of the indicial equation $r(r - 1) + kr - l = 0$. If u is of class $C^{(2)}[0, 1]$ and $f = O(\epsilon)$, then again $\gamma \equiv \epsilon\sigma$, $\sigma \gg 1$, satisfies assumption (8). However, with this choice of a and b, there can exist functions u which only belong to $C^{(2)}(0, 1) \cap C[0, 1]$; as an example, consider the linear problem $(t + \epsilon)^2 y'' + 2(t + \epsilon) y' - y = 0$, $0 < t < 1$. Then the function γ is no longer of order $O(\epsilon)$; instead it satisfies $\gamma \longrightarrow 0$ as $\epsilon \longrightarrow 0^+$, for $t \in [0, 1]$, as follows from the computation in assumption (8). In addition, the boundary layer function is of algebraic type, for

$$\exp\left[-\rho\int_0^t (ba^{-1})(s, \epsilon)ds\right] = \exp\left[rk^{-1}\int_0^t k(s + \epsilon)^{-1}ds\right] = (1 + t\epsilon^{-1})^r.$$

Finally, the result of §2 can be applied to problems in which $f_y(t, y, u'(t), \epsilon)$ is bounded and also problems in which $b(t, \epsilon)$ has a multiple character, for example, $b(t, \epsilon) = k + 2\epsilon(t + \epsilon^2)^{-1}$. Such functions $b(t, \epsilon)$ are briefly discussed in [1] with $a(t, \epsilon) \equiv \epsilon$.

BIBLIOGRAPHY

1. W. Eckhaus, *Matched asymptotic expansions and singular perturbations*, North-Holland, Amsterdam, 1973.

2. L. K. Jackson, *Subfunctions and second-order ordinary differential inequalities*, Advances in Math. 2 (1968), 307–363. MR 37 #5462.

3. R. E. O'Malley, Jr., *Introduction to singular perturbations*, Academic Press, New York, 1974.

4. W. R. Wasow, *Asymptotic expansions for ordinary differential Equations: Trends and problems*, Asymptotic Solutions of Differential Equations: Equations and Their Applications (Proc. Sympos., Math. Res. Center, U. S. Army, Univ. Wisconsin, Madison, Wis., 1964, Wiley, New York, 1964, pp. 3–26. MR 29 #3724.

COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK,UNIVERSITY, NEW YORK, NEW YORK 10012