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MONODROMY GROUPS FOR HIGHER-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we extend certain classical results of Picard and Poincaré (involving monodromy groups of second order equations) to the case of higher-order differential equations.

1. **Preliminaries.** Let F be a compact Riemann surface of genus $g \geqslant 2$ defined by a polynomial equation $P(x, y) = 0$. Use $\xi = (x, y)$ to denote points of F and U to denote the open unit disk. The universal covering map $\pi\colon U \to F$ can then be written $\xi = \pi(t)$, where $\pi(t) = (\varphi(t), \psi(t))$. The associated group of cover transformations will be a Fuchsian group Γ satisfying. $F = U/\Gamma$.

Let E_ξ be any finite subset of F. For reasons of simplicity, we shall assume that E_ξ contains all the branch points of F and all the points situated over $x = \infty$. Let E_t be the pre-image of E_ξ under $\xi = \pi(t)$. We consider two basic linear differential equations (with $n \geqslant 2$):

$$(1) \quad \frac{d^n u}{dx^n} + Q_2(x, y) \frac{d^{n-2}u}{dx^{n-2}} + \cdots + Q_n(x, y)u = 0, \qquad (x, y) \in F;$$

$$(2) \qquad \frac{d^n v}{dt^n} + q_2(t) \frac{d^{n-2}v}{dt^{n-2}} + \cdots + q_n(t)v = 0, \qquad t \in U.$$

It is understood here that the $Q_\alpha(x, y)$ are rational functions and that the $q_\alpha(t)$ are meromorphic on U. We shall say that (1) is of *type* (E_ξ) iff (1) is of Fuchsian type on F and has all its regular singular points located within E_ξ. See, for example, [2, pp. 123, 478]. Similarly (2) is said to be of *type* (E_t, Γ) iff (2) is an equation of Fuchsian type on U, which has all its singular points contained within E_t and which, in addition, is preserved under the change of variable:

$$(3) \quad (t, v) \to (s, w), \quad s = L(t), \quad w = w(s) = v(t)L'(t)^{(n-1)/2}, \qquad L \in \Gamma.$$

It is important to remark that one can choose branches $\xi_L(t)$ of the

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$L'(t)^{1/2}$ so that $\xi_{LK}(t) = \xi_L(Kt)\xi_K(t)$ for $L, K \in \Gamma$ [3, pp. 37–38].

2. **Basic lemmas.**

LEMMA 1. *A one-to-one correspondence between equations of type* (E_ξ) *and* (E_t, Γ) *is induced by*

$$\xi = (x, y) = \pi(t) = (\varphi(t), \psi(t)), \qquad u = u(\xi) = v(t)\varphi'(t)^{(n-1)/2}.$$

This lemma shows that the invariance condition (3) is actually quite natural.

Let us say that equation (2) is of *type* (Γ) whenever the coefficients $q_\alpha(t)$ are holomorphic on U and condition (3) is satisfied.

It is not difficult to give an explicit description of type (Γ) equations for $n = 2, 3, 4$. See, for example, [2, pp. 191–201]. There will in fact be a 1-1 correspondence between such equations and $S_2(\Gamma) \times \cdots \times S_n(\Gamma)$, where $S_k(\Gamma)$ is the vector space of holomorphic k-differentials on U/Γ.

EXAMPLES. For $n = 2$, the defining condition is precisely that $q_2(t) \in S_2(\Gamma)$. For $n = 3$, the condition becomes $q_2(t) \in S_2(\Gamma)$, $q_3(t) - \frac{1}{2}q_2'(t) \in S_3(\Gamma)$. The required 1-1 correspondences are therefore: $q_2 \leftrightarrow q_2$ and $(q_2, q_3) \leftrightarrow (q_2, q_3 - \frac{1}{2}q_2')$.

To give an explicit characterization of this 1-1 correspondence for arbitrary n requires some knowledge of the classical theory of invariants for linear differential equations. One may refer to [2, pp. 191–218], [6, pp. 175–226], and [8, pp. 14–58].

Since $\dim S_k(\Gamma) = (2k - 1)(g - 1)$, the equations (2) of type (Γ) are therefore characterized by $\Sigma_{k=2}^{n}(2k - 1)(g - 1) = (n^2 - 1)(g - 1)$ so-called *accessory parameters.*

LEMMA 2. *Let the column vector* $V(t) = [v_i(t)]$ *consist of n linearly independent solutions of a type* (Γ) *equation. Then*

$$V(Lt) = \chi(L)V(t)\xi_L(t)^{n-1}, \qquad L \in \Gamma, \tag{4}$$

for a uniquely determined homomorphism $\chi\colon \Gamma \longrightarrow \mathrm{SL}(n, \mathbf{C})$.

DEFINITION (By analogy with Poincaré [5]). A holomorphic n-vector $V(t) = [v_i(t)]$ with linearly independent components is said to be a holomorphic zeta-Fuchsian function on U/Γ iff $V(t)$ satisfies equation (4) for some homomorphism $\chi\colon \Gamma \longrightarrow \mathrm{SL}(n, \mathbf{C})$.

LEMMA 3. *Every holomorphic zeta-Fuchsian function on* U/Γ *arises from a uniquely determined equation of type* (Γ).

3. **Main theorems.** The homomorphism χ in (4) is called the *monodromy homomorphism* of $V(t)$. Since $\pi_1(F) \cong \Gamma$, the image group $\chi(\Gamma)$ can be identified with the *monodromy group* of the corresponding differential equations (1) and (2).

THEOREM 1. *If $V_1(t)$ and $V_2(t)$ are holomorphic zeta-Fuchsian functions on U/Γ with the same monodromy homomorphism χ, then $V_1(t) = cV_2(t)$, for some $c \in \mathbf{C}$.*

THEOREM 2. *If $V(t)$ is a holomorphic zeta-Fuchsian function on U/Γ, then its monodromy homomorphism χ must be irreducible as a matrix representation of Γ.*

For $n = 2$, Theorems 1 and 2 are easily reduced to classical assertions of Poincaré and Picard, respectively. See, for example, [**1**, pp. 297, 310] and [**4**, pp. 250–251].

To round out the picture, we mention the following two results which are classical for $n = 2$ [**1**, p. 305] and apparently due to C. Teleman [**7**] for $n \geqslant 3$.

THEOREM 3. *The monodromy homomorphism of a holomorphic zeta-Fuchsian function $V(t)$ can never reduce to a unitary representation.*

COROLLARY. *The monodromy group $\chi(\Gamma)$ of $V(t)$ must therefore be an infinite group.*

Detailed proofs of Theorems 1 and 2 will appear elsewhere.

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