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APPLICATIONS OF DUSCHEK'S FORMULA TO COSMOLOGY AND MINIMAL SURFACES

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1. **The second variation formula.** Let M^{n+1} be a Riemannian or pseudo-Riemannian manifold and let V^n be a compact submanifold (perhaps with boundary) with global unit normal vector field u. We assume u is non-null. Let $\epsilon_u = \langle u, u \rangle = \pm 1$ be its "indicator". If M is pseudo-Riemannian we demand that V be "space-like", i.e. $\epsilon_u = -1$. Consider a smooth 1-parameter variation V_t of V such that each V_t is an embedded submanifold. Each sheet V_t has a unit normal vector field (again called u) and we demand that the variation vector field J always be normal to its sheet, i.e. $J = \varphi u$ for some smooth φ. The first variation of n-volume is classically given by

$$\mathrm{vol}'(t) = -\int_{V_t^n} \varphi H \, dv$$

where $\mathrm{vol}(t) = \mathrm{vol}(V_t^n)$, H is the mean curvature function for V_t, and dv is the volume form.

THEOREM 1. *For second variation we have*

$$\mathrm{vol}''(t) = -\epsilon_u \int_{V_t} \varphi \nabla^2 \varphi \, dv - \int_{V_t} H \frac{\partial \varphi}{\partial t} dv$$

$$+ \int_{V_t} [\mathrm{Ric}(u, u) + \epsilon_u (R_V - R)] \, \varphi^2 dv.$$

Here Ric is the Ricci quadratic form for M, R is the scalar curvature of M, R_V is the scalar curvature of V_t and ∇^2 is the Laplace-Beltrami operator for V_t; both R_V and ∇^2 are constructed from the induced Riemannian metric on V_t.

While this formula is not explicitly given by Duschek in [2] it is certainly implied by other equations appearing there (see his equation (5, 14)). The

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classical Duschek formula involves the second fundamental form b for the hypersurface V_t in the special combination $H^2 - \mathrm{tr}(b^2) = (\mathrm{tr}\, b)^2 - \mathrm{tr}(b^2) = 2\,\Sigma_{\alpha<\beta}\kappa_\alpha\kappa_\beta$, where $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of V_t, i.e. the eigenvalues of b. However, if we write

$$\mathrm{tr}(b \wedge b) = \sum_{\alpha<\beta} \kappa_\alpha \kappa_\beta$$

and use the fact (Gauss equations)

$$-\tfrac{1}{2}\epsilon_u R_V + \mathrm{tr}(b \wedge b) = E(u, u)$$

where $E(u, u) = \mathrm{Ric}(u, u) - \tfrac{1}{2}\epsilon_u R$ is the quadratic form associated with the Einstein tensor, we immediately get Theorem 1.

2. **Minimal submanifolds of a Riemannian M^{n+1}.** V^n is a minimal submanifold of M^{n+1} if $H = 0$.

COROLLARY. *If V^n is a minimal submanifold of a Riemannian M^{n+1} of vanishing Ricci curvature then for normal variations*

$$\mathrm{vol}''(0) = \int_V [\varphi^2 R_V - \varphi \nabla^2 \varphi]\, dv.$$

This generalizes the classical formula (see [1, p. 258]) in the case of a surface $V^2 \subset R^3$, in which case $R_V = 2K$ is twice the Gauss curvature of V.

Recall that a minimal compact submanifold $V^r \subset M^{n+1}$ without boundary is *stable* if $\mathrm{vol}''(0) \geqslant 0$ for *all* variations of V; here vol represents the r-volume of V.

THEOREM 2. *Let M^3 be an orientable 3-manifold with nonpositive sectional curvatures $-b \leqslant K_M \leqslant -c \leqslant 0$. Let V^2 be a closed orientable minimal surface of genus g in M^3. Then if V is stable its area A satisfies*

$$cA/4\pi \leqslant (g-1) \leqslant (3b - c)A/4\pi.$$

This follows from Theorem 1, using the unit normal variation vector $J = u$, and the Gauss-Bonnet theorem.

COROLLARY. *Let M^3 be a compact orientable 3-manifold with strictly negative sectional curvatures $-b \leqslant K_M \leqslant -c < 0$. Then any closed orientable surface (or closed integral current in the sense of Federer and Fleming) of area $< 4\pi(3b - c)^{-1}$ bounds.*

This follows from Theorem 2 and deep results of Federer, Fleming, Almgren, and Lawson, to the effect that any $\alpha \in H_n(M^{n+1}; Z)$, for $n \leqslant 6$, has a representative of *least n*-volume given by a union of closed stable minimal hypersurfaces (see [4, p. L5–45]).

3. **Cosmological expansion.** Consider a space time M^4 filled with a perfect fluid of rest density ρ, pressure p, and unit velocity 4-vector u. Say that p is *spatially constant* if $dp(X) = 0$ for all X orthogonal to u. Let κ be the gravitational constant.

THEOREM 3. *Let M^4 be a space time universe filled with a perfect fluid whose pressure is spatially constant. Suppose that there exists a compact spatial hypersurface V_0^3 (with or without boundary) that is everywhere orthogonal to u. Then the volume* $\mathrm{vol}(t) = \mathrm{vol}(V_t^3)$, *t proper seconds later, of that portion of the fluid initially in V_0^3 satisfies*

$$\mathrm{vol}''(t) = \int_{V_t} [12\pi\kappa(\rho - p) - R_V]\, dv.$$

This follows from Theorem 1, the Einstein equations $E(u, u) = 8\pi\kappa T(u, u)$ relating the Einstein tensor to the stress-energy-momentum tensor, and the fact that the world lines of the fluid are geodesics in M^4 since p is spatially constant.

The fluid is an *incoherent dust* if $p = 0$.

COROLLARY. *If we have an incoherent dust satisfying the hypotheses of Theorem* 3, *then*

$$\mathrm{vol}''(t) = 12\pi\kappa M - \int_{V_t} R_V\, dv$$

where M is the mass of the fluid in V_0. Thus if the spatial universe is initially expanding in volume, then the volume expansion accelerates so long as $\int R_V dv < 12\pi\kappa M$ and decelerates when $\int R_V dv > 12\pi\kappa M$.

This is illustrated by the classical Friedman cosmological models (see [3, pp. 112–125]) which employ spatial sections of spatially constant sectional curvatures. (If one insists on a nonvanishing cosmological constant Λ one should replace $-R_V$ in the above formulas by $3\Lambda - R_V$.)

Details will appear elsewhere.

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