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# LIMITS OF $H^{k,p}$-SPLINES

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Let $I$ be a compact interval in the real line. We denote by $H^{k,p} = H^{k,p}(I)$ $(1 \leqslant p \leqslant \infty,\ k = 1, 2, \ldots)$ the space of real-valued functions which are $k$-fold integrals of functions in $L^p = L^p(I)$. Let $\{t_i\}$, $i = 1, \ldots, n + k$, be chosen in $I$ such that $t_1 \leqslant \cdots \leqslant t_{n+k}$ and $t_{i+k} - t_i > 0$ for $i = 1, \ldots, n$. For the given data $\Gamma = \{\gamma_i, \beta_j\}$, let

$$G_p = \{f \in H^{k,p}\colon [t_i, \ldots, t_{i+k}]f = \gamma_i,\ [t_1, \ldots, t_j]f = \beta_j,\quad 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant k\},$$

where $[x_1, \ldots, x_{r+1}]$ denotes the $r$th divided difference operator at $x_1, \ldots, x_{r+1}$. (This is just another way of writing point evaluations of a function and its derivatives.) For $1 < p < \infty$, let $s_p$ be the unique element in $G_p$ which best approximates the zero element in the seminorm $\|D^k(\cdot)\|_p = \|D^k(\cdot)\|_{L^p(I)}$. The $s_p$'s are called the $H^{k,p}$-splines (which interpolate the given data $\Gamma$). Several authors have studied the $H^{k,p}$-splines. See for example Golomb [5], Jerome and Schumaker [6], and Smith [8]. Mangasarian and Schumaker [7] suggested that an $H^{k,\infty}$-spline could be obtained as a limit of the $H^{k,p}$-splines by taking $p \longrightarrow \infty$. Results in this direction were obtained in [1] and [9]. We now have the following convergence result.

THEOREM 1. *The net* $\{s_p\}_{p>1}$ *converges, as* $p \longrightarrow \infty$, *in* $H^{k,1}$ *to* $s_\infty$ *which is in* $G_\infty$ *and satisfies*

$$\|D^k s_\infty\|_\infty = \inf_{w \in G_\infty} \|D^k w\|_\infty.$$

*Furthermore,* $s_\infty$ *is a* $C^{k-1}(I)$ *(piecewise polynomial) spline or order* $k + 1$ *with no more than* $n$ *knots.*

In fact, we can show that $s_\infty$ is the *Favard solution* [2], [3], and this theorem settles a conjecture of de Boor [2].

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Since the situation when we take $p \longrightarrow \infty$ is so nice, we might expect the convergence of $\{s_p\}_{p>1}$ as $p \longrightarrow 1$. However, one immediate problem is that $G_1$ is not in general proximinal in $H^{k,1}$ under the seminorm $\|D^k(\cdot)\|_1$, and hence there may be no element of minimal seminorm in $G_1$. In order to rectify this situation, we set the problem in $(\mathrm{NBV})^k(I)$, which is the space of functions $f$ whose $k$th derivatives are regular Borel measures $\mu_f$ on $I$ with the total variation seminorm $\|D^{k-1}f\|_{\mathrm{NBV}} = |\mu_f|(I)$. Clearly, $(\mathrm{NBV})^k(I)$ is a dual space and $H^{k,1}$ can be embedded isometrically into $(\mathrm{NBV})^k(I)$. We set

$$\widetilde{G}_1 = \{g \in (\mathrm{NBV})^k(I)\colon [t_i, \ldots, t_{i+k}]g = \gamma_i, [t_1, \ldots, t_j]g = \beta_j, \quad 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k\}.$$

Here we make the simplifying assumption that $t_{i+k-1} > t_i$, since a function in $(\mathrm{NBV})^k(I)$ may not have $(k-1)$th derivatives at the knots. Fisher and Jerome [4] and de Boor [2] have studied the problem of minimizing the $(k-1)$th derivative in NBV. We have the following result.

THEOREM 2. *Every sequence $p_n \longrightarrow 1$ has a subsequence $p'_n$ such that $s_{p'_n} \longrightarrow s$ in the weak* topology of $(\mathrm{NBV})^k(I)$, where $s \in \widetilde{G}_1$ satisfying $\|D^{k-1}s\|_{\mathrm{NBV}} \leqslant \|D^{k-1}g\|_{\mathrm{NBV}}$ for all $g \in \widetilde{G}_1$, and*

$$\|D^{k-1}s\|_{\mathrm{NBV}} = \inf\{\|D^kf\|_1 : f \in G_1\}.$$

In [2] and [4], it was pointed out that there are solutions to the minimum NBV seminorm problem that are $k$-fold integrals of linear combinations of $\delta$-functions. However, we can construct examples to show that the weak* limits of $\{s_p\}_{p>1}$ are *not* necessarily piecewise polynomials. Yet we have the following result.

THEOREM 3. *Let $s \in \widetilde{G}_1$ be a weak* cluster point of $\{s_p\}_{p>1}$ as $p \longrightarrow 1$. Then*

$$D^ks(t) = \sum_{i=1}^{r} c_i\delta(t-\tau_i) + \left(\sum_{i=1}^{m} \pm \chi_{[t_{a_i}, t_{b_i}]}(t)\right)\exp(L(t))$$

*where $L(t) = \Sigma_{i=1}^n d_iN_{i,k}(t)$ is a linear combination of B-splines supported on $[t_1, t_{n+k}]$, $\chi_B(t)$ denotes the characteristic function of the set $B$, and $a_i$, $b_i$ are integers satisfying $1 < a_1 < b_1 < a_2 < b_2 < \cdots < b_m < n + k$. Furthermore,*

$$r + (k-1)m + \sum_{i=1}^{m} (b_i - a_i) \leqslant n.$$

The proofs of the above theorems and more related results will appear elsewhere.

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