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INVARIANTS OF 3-MANIFOLDS

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The μ-invariant $\mu(M)$ of an oriented Z_2-homology 3-sphere M is defined by Hirzebruch in [8], using Rohlin's Theorem [13], to be the mod 16 reduction of the signature of a framed manifold W with $M = \partial W$. In this paper we give a formula for $\mu(M)$ by studying M as a branched dihedral covering space of S^3. Hilden [7] and Montesinos [9] have independently shown that every closed orientable 3-manifold is actually a 3-fold (dihedral) covering space of S^3 branched along a knot.[1] Also see [1], [6] and [12].

Let α be a smooth or piecewise linear oriented knot $S^1 \subset S^3$. Let $V \subset S^3$ with $\partial V = \alpha$ be a Seifert surface for α. The Seifert form $L = L_V$ is the bilinear form of linking numbers of circles in V, with respect to a fixed orientation of S^3, and L' is given as $L'(x, y) = L(y, x)$. Let p be an odd integer. A knot β will be called a mod p *characteristic knot* (in V) of α if there is an embedding of S^1 in V, with nontrivial homology class $[\beta] \in H_1(V)$, so that the composite $S^1 \subset V \subset S^3$ is β; and if $L(x, \beta) + L(\beta, x) \equiv 0 \pmod p$ for all x in $H_1(V)$.

A mod p characteristic knot β for α determines a homomorphism ρ of $\pi_1(S^3 - \alpha)$ onto the dihedral group $Z_2 \times_\omega Z_p$ of order $2p$. The map ρ is characterized by the requirements that its composition with $Z_2 \times_\omega Z_p \longrightarrow Z_2$ be nontrivial and that, for x in the image of $\pi_1 V$, $\rho(x) \in Z_p \subset Z_2 \times_\omega Z_p$ is the mod p reduction of $L(x, \beta)$. Hence β determines a p-fold dihedral branched covering $M_{\alpha,\beta}$ of S^3, branched along α. It can be shown that every dihedral representation for α and associated branched cover of S^3 are determined by a characteristic knot for α in V. Further, dihedral representations can easily be classified in terms of equivalence classes of characteristic knots. By abuse of notation, we write M_α for $M_{\alpha,\beta}$; as "most" knots have at most one (up to conjugacy) dihedral representation of order $2p$, this notation is usually strictly justified.

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[1] In fact one can go directly from a Heegard splitting to a description of any orientable 3-manifold as a dihedral branched covering space.

Let $\alpha_0, \cdots, \alpha_{(p-1)/2}$ be the disjoint oriented circles in M_α that lie over α, with α_0 of branching index 1 and α_i of index 2, $1 \leqslant i \leqslant (p-1)/2$. Orient M_α so that the covering projection has positive degree, and let v_{ij} denote the linking number of α_i with α_j, $i \neq j$. If M_α is a homology sphere, then v_{ij} is an integer, but for a Z_2-homology sphere v_{ij} will in general be a fraction with odd denominator. Let

$$v_{ii} = -\left(\sum_{j=1;j\neq i}^{(p-1)/2} v_{ij} + v_{i0}/2\right), \qquad v_0 = -2 \sum_{i=1}^{(p-1)/2} v_{0i}.$$

Let J be the matrix $((v_{ij}))_{1 \leqslant i,j \leqslant (p-1)/2}$. K. Perko introduced v_0 and has computed v_{ij} and v_0 for many knots.

Let Σ_β be the p-fold cyclic branched cover of S^3, branched along β, and oriented so that the covering projection τ has positive degree. For p a prime-power Σ_β is a rational homology sphere [5].[2] Let T be a covering translation corresponding to a meridian about β. Then

$$\tau^{-1}V = \overline{V} \cup_{\overline{\beta}} T(\overline{V}) \cup_{\overline{\beta}} \cdots \cup_{\overline{\beta}} T^{p-1}(\overline{V})$$

where $\tau | \overline{V}\colon \overline{V} \longrightarrow V$ is a homeomorphism, and where $\overline{\beta} = \tau^{-1}(\beta)$. Let $z_1, \cdots, z_r$ be elements in the image of $H_1(V - \beta)$ in $H_1(V)$ which, together with $[\beta]$, form a basis (over Q) for this image. Let A_i be the matrix whose (j, k)th entry is the linking number in Σ_β of $(\tau|\overline{V})_*^{-1}z_j$ and $T_*^i((\tau|\overline{V})_*^{-1}z_k)$, $1 \leqslant i \leqslant p-1$. Let A have the (j, k)th entry $L_V(z_j, z_k)$. Let $R = ((R_{ij}))$, $1 \leqslant i, j \leqslant (p-1)/2$ be the matrix of blocks where, with subscripts modulo p,

$$R_{ij} = A_{i-j} + A_{j-i} - A_{i+j} - A_{-i-j}, \qquad i \neq j,$$

and

$$R_{ii} = A + A' - 2(A_1 + \cdots + A_{p-1}) - A_{2i} - A_{-2i}.$$

For any knot η in a Z_2-homology sphere, let $\Delta_\eta(t)$ denote its Alexander polynomial.

Let $\hat{\alpha}$ be any knot obtained from $\alpha_1, \cdots, \alpha_{(p-1)/2}$ by connected sum using $(p-3)/2$ paths joining them. (Such paths may be described by lifting suitable paths from α to itself in S^3.)

For any fraction p/q, p and q odd, let $\varphi(p/q) = 0$ if $p/q \equiv \pm 1 \pmod 8$ and $\varphi(p/q) = 8$ if $p/q \equiv \pm 3 \pmod 8$.

[2] When p is not a prime-power and Σ_β not a rational homology sphere, the definition of the matrix R is slightly more complicated.

If N is a Hermitian matrix, let $\sigma(N)$ denote its signature. Let ψ be a primative pth root of unity. Let B be a Seifert matrix for β.

THEOREM. *Suppose that the branched p-fold dihedral covering space M_α of S^3 is a Z_2-homology sphere. Assume that $v_{i,0} \equiv 2 \ (4)$ for $1 \leqslant i \leqslant (p-1)/2$. Then the following holds modulo* 16:

$$\mu(M_\alpha) = \sum_{i=1}^{p-1} \sigma(B + B' - B\psi^i - B'\psi^{-i}) + \left(\frac{p-1}{2}\right)\varphi\ (\Delta_\alpha(-1))$$

$$+ \varphi(p)L_V([\beta], [\beta]) + \varphi(\Delta_{\hat{\alpha}}(-1)/\Delta_{\hat{\alpha}}(1)) - (v_0/4) + \sigma(J) - \sigma(R).$$

The terms on the right of this formula are readily calculable. Note that some of the terms vary with the choice of characteristic knot β. (However, for suitable dihedral covers of ribbon knots these terms contribute zero. This gives a simple obstruction to a knot being a ribbon knot.) For a homology sphere M_α, the terms on the right are integers. For "bushel baskets" [6] of knots α, $\pi_1(M_\alpha) = 0$.[3]
some of the terms vary with the choice of characteristic knot β. For a homology sphere M_α, the terms on the right are integers. For "bushel baskets" [6] of knots α, $\pi_1(M_\alpha) = 0$.[3]

The condition on $v_{i,0}$ seems to be satisfied in all known cases for M_α a Z_2-homology sphere [10], [11]. It implies that $\det J \neq 0$, which suffices for the theorem. Therefore it seems reasonable to conjecture at least that $\det J \neq 0$ if M_α is a Z_2-homology sphere. For $p = 3$, we can show that $v_{1,0} = 2$ (mod 4). (See [2] and [10] for 2-bridge knots.) Hence the Theorem applies in this case. By [7] and [9] the case $p = 3$ of our formula applies to every Z_2-homology sphere.

REFERENCES

1. J. W. Alexander, *Note on Riemann spaces,* Bull. Amer. Math. Soc. 26 (1919), 370–372.

2. C. Bankwitz and H. G. Schumann, *Über Viergeflechte,* Abh. Math. Sem. Univ. Hamburg 70 (1934), 263–284.

3. S. E. Cappell and J. L. Shaneson, *Cyclic branched covering spaces* Ann. of Math. Studies (Memorial Volume in honor of R. Fox).

4. A. H. Clifford, *On the canonical form and dissection of a Riemann's surface,* Proc. London Math. Soc. 8 (1877), 292–304.

[3] For M a homotopy 3-sphere, $\mu(M) = 0$ if $M \times S^1 \times S^1$ is P.l. homeomorphic to $S^3 \times S^1 \times S^1$ and $\mu(M) \neq 0$ if $M \times S^1 \times S^1$ is P.l. homeomorphic to the exotic manifold described in [14] which is homotopy equivalent but not P.l. homeomorphic to $S^3 \times S^1 \times S^1$.

5. R. H. Fox, *A quick trip through knot theory,* Topology of 3-Manifolds and Related Topics (Proc. Univ. of Georgia Inst., 1961), Prentice-Hall, Englewood Cliffs, N. J., 1962, pp. 120–167. MR 25 #3522.

6. ———, *Construction of simply connected 3-manifolds,* Topology of 3-Manifolds and Related Topics (Proc. Univ. of Georgia Inst., 1961), Prentice-Hall, Englewood Cliffs, N. J., 1962, pp. 213–216. MR 25 #3539.

7. H. Hilden, *Every closed orientable 3-manifold is a 3-fold branched covering space of* S^3, Bull. Amer. Math. Soc. 80 (1974), 1243–1244.

8. F. Hirzebruch, W. P. Newmann and S. S. Koh, *Differentiable manifolds and quadratic forms,* Dekker, 1971.

9. M. Montesinos, *A representation of closed, orientable 3-manifolds as 3-fold branched coverings of* S^3, Bull. Amer. Math. Soc. 80 (1974), 845–846.

10. K. Perko, *On dihedral covering spaces of knots* (to appear).

11. ———, Unpublished tables available upon request to the author, 1 Chase Manhattan Plaza, New York, N. Y.

12. K. Reidemeister, *Knoten Theorie*, Springer-Verlag, 1932.

13. V. A. Rohlin, *New results in the theory of four-dimensional manifolds,* Dokl. Akad. Nauk SSSR 84 (1952), 221–224. (Russian) MR 14, 573.

14. J. L. Shaneson, *Wall's surgery obstruction groups for* $G \times Z$, Ann of Math. (2) 90 (1969), 296–334. MR 39 #7614.

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