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MATRIX DIFFERENTIAL EQUATIONS

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Let Ω be the set of n by n matrices with complex elements, let R denote the set of reals, and let R_0 denote the interval $[0, t_0)$ for some $t_0 > 0$. We consider the differential relation

$$0 \in z' - f(t, z), \qquad t \in R_0 \tag{1}$$

where $z(t) \in \Omega$ and f is a function from $R_0 \times \Omega$ to subsets of Ω. The equation can be interpreted in two senses: Either z is absolutely continuous and the relation holds almost everywhere, or z is continuous and the relation holds except in a countable set.

A function $\phi(t, \rho)$ from $R_0 \times R$ to R is a uniqueness function if the upper solution of the equation

$$D^+\rho = \phi(t, \rho), \qquad t \in R_0; \quad \rho(0) = 0 \tag{2}$$

is $\rho = 0$. Here D^+ denotes the upper right Dini derivate, though other derivates could be used just as well. The equation (2) is interpreted in the same sense as (1).

We use $|\xi|$ for the Euclidean length of the complex vector ξ, so that $|\xi|^2 = \xi^*\xi$. For $z \in \Omega$ a norm and Kamke norm are defined respectively by

$$\|z\| = \sup|z\xi|, \qquad [z] = \sup \operatorname{Re}(\xi^* z\xi), \quad (|\xi| = 1).$$

We say that f satisfies a uniqueness condition if there exist an $\epsilon > 0$ and a uniqueness function ϕ such that

$$x \in f(t, u), \qquad y \in f(t, v), \qquad \|u - v\| < \epsilon$$

together imply

$$[(u - v)^*(x - y)] \leqslant \|u - v\|\phi(t, \|u - v\|).$$

The hypotheses and conclusions of our theorems hold for $t \in R_0$ and, for simplicity, all coefficients in the examples are integrable.

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Invariance of the unit ball. If $\|z(0)\| \leqslant 1$ implies $\|z(t)\| \leqslant 1$ for solutions of (1), it is said that the unit ball is invariant. This is the case (and the proof is easy) if there exists $\epsilon > 0$ such that the three conditions

$$1 + \epsilon > \|v\| > 1, \qquad w \in f(t, v), \qquad v = z(t)$$

imply $[v^*w] \leqslant 0$. The following theorem has a uniqueness requirement on f, but in other respects the hypothesis is much weaker than that above.

THEOREM 1. *Let f satisfy a uniqueness condition. Suppose further that there is some $w \in f(t, v)$ such that the three conditions*

$$\|v\| = 1, \qquad \eta = v\xi, \qquad \xi = v^*\eta$$

together imply $\mathrm{Re}(\eta^* w \xi) \leqslant 0$. *Then the unit ball is invariant.*

The uniqueness condition is needed only relative to points v on $\|v\| = 1$ and u on $1 < \|u\| < 1 + \epsilon$ with $u = z(t)$. For proof, it is sufficient to show that

$$\liminf_{\alpha \to 0^+} \frac{\|v + \alpha w\| - \|v\|}{\alpha} \leqslant 0 \tag{3}$$

and the result follows from known invariance theorems [1], [2], [3], [5] as extended in [6]. Because of its relation to (3), the hypothesis $\mathrm{Re}(\eta^* w \xi) \leqslant 0$ under the restrictions of Theorem 1 is referred to as the *tangent condition* on $\|z\| = 1$.

For example, let z satisfy the Riccati equation $z' = a + bz + zd + zcz$. Theorem 1 immediately gives a result of Reid [7] and the author, to the effect that the unit ball is invariant if

$$\mathrm{Re}(\eta^* a \xi + \eta^* b \eta + \xi^* d \xi + \xi^* c \eta) \leqslant 0 \quad \text{for } |\xi| = |\eta|. \tag{4}$$

It is seen, incidentally, that the Riccati equation satisfies the tangent condition for $\|z\| = 1$ if it satisfies the tangent condition for $z^*z = 1$.

Theorem 1 applies with equal ease to equations of higher degree. For instance let

$$z' = a_1 + b_1 z + z d_1 + z c_1 z + z z^* a_2 z^* z$$
$$+ z z^* b_2 z + z d_2 z^* z + z z^* z c_2 z z^* z$$

where the coefficients are integrable. Then invariance of the unit ball follows from (4) with

$$a = a_1 + a_2, \quad b = b_1 + b_2, \quad c = c_1 + c_2, \quad d = d_1 + d_2.$$

Invariance of the order cone. We use the order induced by quadratic forms, so $z \leqslant 0$ holds if and only if $[z] \leqslant 0$. Thus $[z]$ is one of the Kamke norms which generate the order relation [4]. The order cone is said to be invariant if $z(0) \geqslant 0$ implies $z(t) \geqslant 0$ for $t \in R_0$.

THEOREM 2. *Let f satisfy a uniqueness condition. Suppose further that there is some $w \in f(t, v)$ such that the two conditions*

$$v + v^* \geqslant 0, \qquad (v + v^*)\xi = 0$$

together imply $\xi^*(w + w^*)\xi \geqslant 0$. Then the order cone is invariant.

For proof, note that the functionals generating the order cone are of the form ϕ_ξ, where $\phi_\xi(z) = \mathrm{Re}(\xi^* z \xi)$ and $|\xi| = 1$. It is possible to show that the quasimonotony condition of Volkmann [8] holds under the hypothesis of Theorem 2; hence by [5] the tangent condition holds on the order cone; and Theorem 2 follows from [1], [2], [3], [5] as extended in [6]. For single-valued functions Theorem 2 also follows from [4], Theorem 2; note that $[p] = 1$, $[-p] = -1$ where p is the identity matrix.

Here again, v and w are somewhat more restricted than stated in the theorem. In particular if the differential equation is Hermitian, so that the solution satisfies $z = z^*$ and $f(z, t) = f(z, t)^*$, then the hypothesis is needed only when $v = v^*$ and $w = w^*$.

As a simple illustration let $z' \geqslant bz + zb^* + zcz + zcz^*$. Then $z(0) \geqslant 0$ implies $z(t) \geqslant 0$; compare Reid [7]. If $Tu = u' - g(t, u)$ monotony in the sense of Collatz can be deduced by applying Theorem 2 with $f(t, z) = g(t, u + z) - g(t, u)$. For example let $Tu = u' - ub - b^*u - ucu$ where $u^*c^* = uc$. Then $Tu \leqslant Tv$ and $u(0) \leqslant v(0)$ implies $u(t) \leqslant v(t)$. Note that the choice $g(t, u) = u^2$ is permissible here though specifically excluded in [7, Theorem 6.1]. All these applications of Theorem 2 extend to equations of higher degree and to the comparison of two operators, as in $T_1 u \leqslant T_2 v$.

Cayley transforms and periodic solutions. It is not difficult to show that the transformation

$$w = (z - 1)(z + 1)^{-1}, \qquad z = (1 + w)(1 - w)^{-1}$$

effects a formal conversion of either Theorem 1 or 2 into the other. Thus the two theorems can be regarded as being in reality a single one, even though they have their roots in two rather distinct historical traditions. This observation gives a unified approach to a substantial and diverse literature.

The Cayley transform also has an interesting bearing on periodic solutions. Let $f(z, t)$ have t-period ω. If the unit ball is invariant then, subject to mild continuity conditions, the transformation $z(0) \longrightarrow z(\omega)$ has a fixed point in $\|z\| \leqslant 1$, and a periodic solution exists. A slightly sharper hypothesis ensures $\|z(t)\| < 1$ for $t > 0$ in Theorem 1, so that the fixed point is also in $\|z\| < 1$; this is sometimes desirable for technical reasons. In the case of a Riccati equation the transformation $z(0) \longrightarrow z(\omega)$ is a linear fractional transformation, the theory of fixed points is simpler than in the general case, and the argument extends to equations in which the unknowns are operators on a Hilbert space.

This discussion gives existence of a periodic solution under conditions in which the main hypothesis is that of Theorem 1. However the argument does not apply to Theorem 2, because the set $z \geqslant 0$ is not compact even in the finite-dimensional case. But by a Cayley transformation $z \longrightarrow w$ we can get a periodic solution w from Theorem 1, and transforming back gives a periodic z. Hence, there are theorems asserting existence of a periodic solution in which the main hypothesis is that of Theorem 2.

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