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A UNIQUENESS THEOREM FOR HOMOLOGY IN $\mathcal{C}$at, THE CATEGORY OF SMALL CATEGORIES

BY DANA MAY LATCH

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I. **Introduction.** Oberst [7], Laudal [4], Watts [10], and André [1] have shown that derived functors of colimit define a homology theory for $\mathcal{C}$at, the category of small categories. In this note, we outline a proof of uniqueness for such a homology theory, making extensive use of a Kan-type construction (see e.g. Lemma A) and of uniqueness for homology in $\mathcal{S}^{\Delta^{\mathrm{op}}}$, the category of simplicial sets [2].

II. **Preliminaries.** The following Kan-type construction is used in several contexts.

LEMMA A. *Let $\mathcal{C}$ be a cocomplete category, $\mathbf{C}$ a small category, and θ: $\mathbf{C} \longrightarrow \mathcal{C}$ a functor. Then there exists an adjoint pair: the singular functor S_θ: $\mathcal{C} \longrightarrow \mathcal{S}^{\mathbf{C}^{\mathrm{op}}}$ defined by $S_\theta(A) = \mathcal{C}(\theta\_, A)$, for $A \in |\mathcal{C}|$, and its left adjoint $\hat{\theta}$: $\mathcal{S}^{\mathbf{C}^{\mathrm{op}}} \longrightarrow \mathcal{C}$.*

Let Δ be the small category whose objects are the finite ordinals $[k] = \{0 < 1 < 2 < \ldots < k\}$ and whose morphisms are order preserving functions μ: $[k] \longrightarrow [m]$. By considering the full inclusion functor ι: $\Delta \longrightarrow \mathcal{C}$at, in the context of Lemma A, nerve, N: $\mathcal{C}$at $\longrightarrow \mathcal{S}^{\Delta^{\mathrm{op}}}$, is the singular adjoint of categorical realization c: $\mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{C}$at and $cN = \mathrm{id}_{\mathcal{C}\mathrm{at}}$ [3, p. 33]. Thus the standard representable k-dimensional simplicial set $\Delta[k]$ is actually $N([k]) = \Delta(-, [k])$.

Similarly, the functor τ: $\Delta \longrightarrow \mathcal{C}$at defined as the comma category, $\tau[k] = \Delta \downarrow [k]$, gives rise to another pair of adjoint functors S: $\mathcal{C}$at $\longrightarrow \mathcal{S}^{\Delta^{\mathrm{op}}}$ and Γ: $\mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{C}$at. Let $X \in |\mathcal{S}^{\Delta^{\mathrm{op}}}|$, then ΓX is the small category whose objects are $\amalg_{k \geq 0} X_k$, and whose morphisms are triples $\langle y, \mu, x\rangle$ where $x \in X_m$ is the codomain, μ: $[k] \longrightarrow [m]$ in Δ is the morphism, and $y = X(\mu)x$ in X_k is the domain.

The natural transformation "last", η: $\tau \xrightarrow{\cdot} \iota$, is given by $\eta_k(\alpha$: $[p] \longrightarrow [k]) = \alpha(p) \in [k]$. By adjoint functor theory and by the theory of coends, η induces natural transformations $\eta^1: N \xrightarrow{\cdot} S$, $\eta^2: \Gamma \xrightarrow{\cdot} c$, $\eta^3: \Gamma N \xrightarrow{\cdot} cNx = \mathrm{id}_{\mathcal{C}\mathrm{at}}$, and $\eta^4: N\Gamma \xrightarrow{\cdot} \mathrm{id}_{\mathcal{S}^{\Delta^{\mathrm{op}}}}$.

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The Milnor geometric realization functor $|\ |: \mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{T}op$ [5], $\mathcal{T}op$ the category of CW complexes, can also be viewed as another example of the "Lemma A" situation. If $\theta: \Delta \longrightarrow \mathcal{T}op$ is given by $\theta([k]) = \Delta^k$, the standard k-dimensional affine simplex in $\mathbf{R}^{k+1}$, then $|\ |: \mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{T}op$ is the left adjoint of the singular complex functor $S_\theta: \mathcal{T}op \longrightarrow \mathcal{S}^{\Delta^{\mathrm{op}}}$. Let $B = |N\_|: \mathcal{C}at \longrightarrow \mathcal{T}op$ denote the Segal classifying space functor [9]. Then for each small category $\mathbf{C}$, $B\mathbf{C}$ is the CW complex whose k-cells are in one-to-one correspondence with nondegenerate k-simplicies of $N\mathbf{C}$.

III. **Definition of homology and existence.** A subcategory $\mathbf{C}'$ of $\mathbf{C}$ is *initial* in $\mathbf{C}$ if all morphisms $m: p \longrightarrow q$ in $\mathbf{C}$ with codomain in $\mathbf{C}'$ are in $\mathbf{C}'$. A pair of small categories $(\mathbf{C}, \mathbf{C}')$ is said to be *admissible* if $\mathbf{C}'$ is an initial subcategory of $\mathbf{C}$. The category of all admissible pairs and obvious morphisms is also denoted by $\mathcal{C}at$.

A *homology theory* for $\mathcal{C}at$ is a pair $\langle h, \partial\rangle$, where $h: \mathcal{C}at \longrightarrow \mathcal{A}b^{\mathbf{Z}}$ is a functor from the category of admissible pairs to graded abelian groups and $\partial_*: h_* \longrightarrow h_{*-1}$ is a natural transformation of degree -1, satisfying the standard Eilenberg-Steenrod-Milnor axioms [6]: dimension, exactness, excision, homotopy and strong additivity. We state the last three below.

Excision axiom. Let $\mathbf{C}$ be any small category with initial subcategories $\mathbf{C}_1$ and $\mathbf{C}_2$. Then the inclusions induce $h_*(\mathbf{C}_2, \mathbf{C}_1 \cap \mathbf{C}_2) \cong h_*(\mathbf{C}_1 \cup \mathbf{C}_2, \mathbf{C}_1)$, where $\mathbf{C}_1 \cap \mathbf{C}_2$ and $\mathbf{C}_1 \cup \mathbf{C}_2$ are subcategories of $\mathbf{C}$ making the following square bicartesian in $\mathcal{C}at$:

$$\begin{tikzcd} \mathbf{C}_1 \cap \mathbf{C}_2 \arrow[r] \arrow[d] & \mathbf{C}_1 \arrow[d] \\ \mathbf{C}_2 \arrow[r] & \mathbf{C}_1 \cup \mathbf{C}_2. \end{tikzcd}$$

Homotopy axiom. If $F: \mathbf{C} \longrightarrow \mathbf{D}$ is a *weak homotopy* equivalence, i.e. if $BF: B\mathbf{C} \longrightarrow B\mathbf{D}$ is a homotopy equivalence in $\mathcal{T}op$, then $h_*F: h_*\mathbf{C} \longrightarrow h_*\mathbf{D}$ is an isomorphism.

Strong additivity (Milnor) axiom. Let $\{(\mathbf{C}_\alpha, \mathbf{C}'_\alpha) \mid \alpha \in A\}$ be a collection of admissible pairs in $\mathcal{C}at$. Then the inclusions induce

$$\bigoplus_\alpha h_*(\mathbf{C}_\alpha, \mathbf{C}'_\alpha) \cong h_*\Big(\coprod_\alpha \mathbf{C}_\alpha, \coprod_\alpha \mathbf{C}'_\alpha\Big).$$

We assume that the coefficient group $A \in \mathcal{A}b$ is fixed. More general coefficient systems will be discussed in the longer exposition.

Define $\Delta_{\mathbf{C},\mathbf{C}'}(A): \mathbf{C} \longrightarrow \mathcal{A}b$ by

$$\Delta_{\mathbf{C},\mathbf{C}'}(A)_p = \begin{cases} A & \text{if } p \notin |\mathbf{C}'|, \\ 0 & \text{otherwise} \end{cases}$$

on objects and in the obvious fashion on morphisms.

REMARK. $\mathbf{C}'$ initial in $\mathbf{C}$ guarantees that $\Delta_{\mathbf{C},\mathbf{C}'}(A)$ is a functor. Other pairs, e.g., $\mathbf{C}'$ terminal in $\mathbf{C}$ or $\mathbf{C}'$ an "interval" in $\mathbf{C}$ would also satisfy this condition.

THEOREM 1. *$\langle H, \partial\rangle$ is a homology theory for* Cat, *where*

$$H_*(\mathbf{C}, \mathbf{C}') = L_* \operatorname*{colim}_{\mathbf{C}}(\Delta_{\mathbf{C},\mathbf{C}'}(A)),$$

$L_* \operatorname{colim}:_{\mathbf{C}} \mathsf{Ab}^{\mathbf{C}} \longrightarrow \mathsf{Ab}$ *being the left derived functors of* $\operatorname{colim}_{\mathbf{C}}: \mathsf{Ab}^{\mathbf{C}} \longrightarrow \mathsf{Ab}$.

PROOF. See [7] and [4].

Using the canonical coflabby resolution of $\Delta_{\mathbf{C},\mathbf{C}'}(A)$ ([**10**], [4], [7]) we see that the complexes used to calculate the homology yield the following:

COROLLARY. *Let $\langle H, \partial\rangle$ be the unique homology in $S^{\Delta^{\mathrm{op}}}$* [2], *i.e. singular homology. Then*

$$\begin{tikzcd} \mathsf{Cat} \arrow[rr, "N"] \arrow[dr, "H_*"'] & & S^{\Delta^{\mathrm{op}}} \arrow[dl, "H_*"] \\ & \mathsf{Ab}^Z & \end{tikzcd}$$

commutes up to isomorphism.

IV. **Uniqueness.** The proof of our uniqueness theorem rests on the following two lemmas both of which are used in applying the homotopy axiom.

LEMMA B. *The natural transformation η^4: $N\Gamma \overset{\cdot}{\longrightarrow} \mathrm{id}_{S^{\Delta^{\mathrm{op}}}}$ induces a natural homotopy equivalence $|\eta^4_X|$: $|N\Gamma X| \longrightarrow |X|$ in* Top, *for every simplical set X.*

LEMMA C. *The natural transformation η^3: $\Gamma N \overset{\cdot}{\longrightarrow} \mathrm{id}_{\mathsf{Cat}}$ is a weak homotopy equivalence in* Cat, *i.e. η^3: $\Gamma N\mathbf{C} \longrightarrow \mathbf{C}$ is a weak homotopy equivalence for each small category* $\mathbf{C}$.

THEOREM 2. *If $\langle h, \partial\rangle$ is a homology theory for* Cat, *then*

$$\begin{tikzcd} S^{\Delta^{\mathrm{op}}} \arrow[rr, "\Gamma"] \arrow[dr, "H_*"'] & & \mathsf{Cat} \arrow[dl, "h_*"] \\ & \mathsf{Ab}^Z & \end{tikzcd}$$

commutes up to isomorphism.

The proof consists of showing that $h_*\Gamma\colon \mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{A}b^Z$ satisfies the standard axioms for homology theory in $\mathcal{S}^{\Delta^{\mathrm{op}}}$. Hence, by uniqueness of such a theory [2], the diagram commutes. Some of the special properties of $\Gamma\colon \mathcal{S}^{\Delta^{\mathrm{op}}} \longrightarrow \mathcal{C}at$ used in the proof are that Γ commutes with pullbacks, $\Gamma(\Delta[k])$ is contractible, and $(\Gamma X, \Gamma X') = \Gamma(X, X')$ is an admissible pair. Lemma B is needed in proving the homotopy axiom.

THEOREM 3 (UNIQUENESS). *If $\langle h, \partial\rangle$ is a homology theory for $\mathcal{C}at$ then* $h_*(\mathbf{C}, \mathbf{C}') \cong H_*(\mathbf{C}, \mathbf{C}')$.

PROOF. $h_*(\mathbf{C}, \mathbf{C}') \cong h_*(\Gamma N\mathbf{C}, \Gamma N\mathbf{C}')$ by the homotopy axiom used in conjunction with Lemma C. By Theorem 2, $h_*(\Gamma N\mathbf{C}, \Gamma N\mathbf{C}') \cong H_*(N\mathbf{C}, N\mathbf{C}')$. But the Corollary guarantees that $H_*(N\mathbf{C}, N\mathbf{C}') \cong H_*(\mathbf{C}, \mathbf{C}')$. Q.E.D.

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DEPARTMENT OF MATHEMATICS, RUTGERS UNIVERSITY, NEW BRUNSWICK, NEW JERSEY 08903

Current address: Department of Mathematics, Lawrence University, Appleton, Wisconsin 54911