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CONICAL DISTRIBUTIONS FOR RANK ONE SYMMETRIC SPACES

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Let $X = G/K$ be a symmetric space of noncompact type, where G is a connected semisimple Lie group with finite center, and K is the compact part of an Iwasawa decomposition $G = KAN$ of G. Let $M\,(M')$ be the centralizer (normalizer) of A in K. Then the space Ξ of all horocycles of X can be identified with G/MN or $(K/M) \times A$ [1, p. 8]. The set of all smooth functions with compact supports on Ξ endowed with the customary topology is denoted by $\mathcal{D}(\Xi)$. Its dual $\mathcal{D}'(\Xi)$ consists of all distributions on Ξ. Let W be the Weyl group M'/M and $\mathfrak{A}_C^*$ be the complex dual of $\mathfrak{A}$, the Lie algebra of A.

DEFINITION [1, p. 65]. A distribution $\Psi \in \mathcal{D}'(\Xi)$ is said to be *conical* if (i) Ψ is MN-invariant, (ii) Ψ is an eigendistribution of every G-invariant differential operator on Ξ.

As is readily seen, this definition is parallel to that of spherical functions on X. On this basis S. Helgason made the conjecture that the set of all conical distributions can be parametrized by $W \times \mathfrak{A}_C^*$, and he also established it in various cases [1, Chapter III, §4]. Our purpose here is to complete its verification in case X has rank one.

Now for each $a \in A$, there is a map $\sigma(a)$ of Ξ defined by $\sigma(a)(gMN) = gaMN$. This gives rise to a corresponding action $\Psi \longmapsto \Psi^{\sigma(a)}$ on the space $\mathcal{D}'(\Xi)$. If $\lambda \in \mathfrak{A}_C^*$, let $\mathcal{D}'_\lambda = \{\Psi \in \mathcal{D}'(\Xi) \mid \Psi^{\sigma(a)} = e^{-(i\lambda+\rho)\log a}\Psi,\ \forall a \in A\}$, where ρ is half the sum of all positive restricted roots, counting multiplicity, and $\log : A \longrightarrow \mathfrak{A}$ is the inverse of the exponential map. The space $\mathcal{D}'_\lambda$ consists of the joint eigenspaces of the G-invariant differential operators on Ξ [1, p. 69]. So an element $\Psi \in \mathcal{D}'(\Xi)$ is conical iff it is (i) MN-invariant, and (ii) belongs to some $\mathcal{D}'_\lambda$. Next we recall some constructions from [1, Chapter III, §4]. For each $s \in M'/M$, choose an $m_s \in M'$ in the

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coset s and put $\xi_s = m_s MN$, $\Xi_s = MNA\,\xi_s$. The group MNA induces a regular Borel measure $d\nu_s$ on Ξ_s which is invariant under MN and $\sigma(a)$, $a \in A$. For $\xi \in \Xi_s$, let $a(\xi)$ be the unique element in A such that $\xi \in MNa(\xi)\xi_s$. Let $\langle\ ,\ \rangle$ be the Killing form, Σ^+ the set of all positive restricted roots, $\Sigma^- = -\Sigma^+$, and Σ_0^+, the set of all indivisible roots in Σ^+.

THEOREM 1 [1, p. 82]. *If* $\lambda \in \mathfrak{A}_C^*$ *satisfies* $\mathrm{Re}\langle \alpha, i\lambda\rangle > 0$ *for all* α *in* $\Sigma^+ \cap s^{-1}\Sigma^-$, *then the linear functional*

$$\Psi'_{\lambda,s}\colon \phi \longmapsto \int_{\Xi} \phi(\xi)\exp\left[(is\lambda + s\rho)(\log a(\xi))\right] d\nu_s, \quad \phi \in \mathcal{D}(\Xi),$$

is a conical distribution in $\mathcal{D}'_\lambda$.

THEOREM 2 [1, p. 88]. *Let*

$$d_s(\lambda) = \prod_{\alpha\in\Sigma_0^+\cap s^{-1}\Sigma_0^-} \Gamma\left(\frac{\langle i\lambda, \alpha\rangle}{\langle \alpha, \alpha\rangle}\right), \qquad \lambda \in \mathfrak{A}_C^*.$$

Then the map $\lambda \longmapsto d_s^{-1}(\lambda)\Psi'_{\lambda,s}$ *extends from the tube* $\{\lambda \,|\, \mathrm{Re}\langle i\lambda, \alpha\rangle > 0$ *for all* $\alpha \in \Sigma^+ \cap s^{-1}\Sigma^-\}$ *to a distribution valued holomorphic function* $\Psi_{\lambda,s}$ *on* $\mathfrak{A}_C^*$. *For each* λ, $\Psi_{\lambda,s}$ *is a conical distribution in* $\mathcal{D}'_\lambda$.

Assume, in the sequel, that the rank of X equals 1. Let $s \in W$ be the nontrivial element and $e \in W$ the identity. Let α be the element in Σ_0^+, and m_α the multiplicity of α. Let $d\xi$ be the G-invariant measure on Ξ.

It is noted in [1] that if $\lambda = 0$, all the $\Psi_{\lambda,s}$, $s \in W$, constructed in Theorem 2 are proportional. The distribution Ψ_0 in Theorem 3 provides a compensation for this.

THEOREM 3. *For* $\phi \in \mathcal{D}(\Xi)$, *let* $\phi_0 \in \mathcal{D}(\Xi)$ *be given by* $\phi_0(kaMN) = \phi(aMN)$. *Then*

$$\Psi_0\colon \phi \longmapsto \int_{\Xi} (\phi(\xi) - \phi_0(\xi))e^{\rho(\log a(\xi))}\,d\xi$$

is a conical distribution in $\mathcal{D}'_0$.

From the construction we see easily that $\Psi_{\lambda,e}$ is concentrated on $\Xi_e = AMN$. So is $\Psi_{\lambda,s}$ if $-i\lambda$ is a positive integral multiple of α. Conversely, we have

THEOREM 4. *Assume* $m_\alpha \neq 1$. *If the conical distribution* $\Psi \in \mathcal{D}'_\lambda$ *is* *concentrated on* Ξ_e *and not proportional to* $\Psi_{\lambda,e}$, *then* $-i\lambda$ *is a positive integral multiple of* α, *and* Ψ *is a linear combination of* $\Psi_{\lambda,e}$ *and* $\Psi_{\lambda,s}$. (*For* $G = SO_0(1, n), n > 2$, *cf.* [1, *Chapter* III, *Theorem* 4.10].)

With some more technical lemmas and the aid of Theorem 4.9 in [1, Chapter III], we finally arrive at the following main result:

THEOREM 5. *Assume the symmetric space* G/K *has rank* 1 *and dimension* > 2. *Let* $\Psi \in \mathcal{D}'_\lambda$ *be conical. We have*

(i) *if* $\lambda = 0$, *then* $\Psi = c\Psi_0 + c'\Psi_{\lambda,e}$;

(ii) *if* $\lambda \neq 0$, *then* $\Psi = c\Psi_{\lambda,s} + c'\Psi_{\lambda,e}$,

where c *and* c' *are constants.*

In case the dimension of $G/K = 2$, there is one more base element for the conical distributions in $\mathcal{D}'_\lambda$ if $i\lambda = (\frac{1}{2} - l)\alpha$, l being a positive integer. This discrepancy disappears, however, if we modify the definition of conical distributions so that G is the whole (not necessarily connected) isometry group. After this modification, Theorem 5 is valid for all rank one spaces. In this sense, Helgason's conjecture is true for all rank one symmetric spaces.

REFERENCES

1. S. Helgason, *A duality for symmetric spaces with applications to group representations*, Advances in Math. 5 (1970), 1–154. MR 41 #8587.

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