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HOMOTOPY TREES FOR PERIODIC GROUPS

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Let π be a finite periodic group of order n whose cohomology has minimal period k. We say that π has *free period* h if π admits a periodic free resolution of the trivial π-module $\mathbf{Z}$ of length h:

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathbf{Z}\pi \longrightarrow C_{h-2} \longrightarrow C_{h-3} \longrightarrow \cdots \longrightarrow C_1 \longrightarrow \mathbf{Z}\pi \xrightarrow{\epsilon} \mathbf{Z} \longrightarrow 0$$

where each C_i is a finitely-generated free π-module. Acccording to [7], every finite periodic group of minimal period k has a minimal free period $h = pk$ for some integer $p > 0$. A convenient listing of all finite periodic groups is given in [9].

DEFINITION. A (π, m)*-complex* is a finite, connected m-dimensional CW complex X with fundamental group π whose universal cover $\widetilde{X}$ is $(m-1)$-connected.

Let $HT(\pi, m)$ denote the set of homotopy types of (π, m)-complexes. This set may be described as a *directed tree* with one vertex for each homotopy class $[X]$ of (π, m)-complexes having the homotopy type of X; the vertex $[X]$ is connected by an edge to vertex $[Y]$ provided Y has the homotopy type of the sum $X \vee S^m$ of X with the m-sphere S^m. $HT(\pi, m)$ is *connected* by [11, Theorem 14] and clearly contains no circuits.

The purpose of this note is to announce a complete description of the homotopy tree $HT(\pi, m)$ for certain periodic π and for $m = ik, ik - 1$ $(i > 0)$. Full details and a description for any periodic π will appear elsewhere.

Before stating the theorem, we need two more pieces of notation. Let $\mathbf{Z}_n^*$ be the units of the ring $\mathbf{Z}_n$ of integers modulo n. Then $\operatorname{Aut}_k \pi = \{p \in \mathbf{Z}_n^* \mid \exists \alpha \in \operatorname{Aut} \pi \ni \alpha_k^*(1) = p$ where $\alpha_k^*: H^k(\pi, \mathbf{Z}) \longrightarrow H^k(\pi, \mathbf{Z})\}$. Let $\widetilde{K}_0\mathbf{Z}\pi$ be the reduced projective class group of the integral group ring $\mathbf{Z}\pi$ of π. Define a homomorphism $\nu: \mathbf{Z}_n^* \longrightarrow \widetilde{K}_0\mathbf{Z}\pi$ by $\nu(p) =$ class of the projective left ideal (p', N) of $\mathbf{Z}\pi$ generated by any integer $p' \in p$ and $N = \Sigma_{x \in \pi} x$. ν is well defined by [7, Lemma 6.1].

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THEOREM 1. *Let* π *be a finite periodic group of order* n *and minimal free period* k. *Furthermore, suppose that* π *is either abelian or that* 8 *does not divide the order of* π. *Let* R_m *be a* (π, m)*-complex of minimal absolute Euler characteristic.*

(a) *Let* $m = ik$ $(i > 0)$ *and* X *be any* (π, ik)*-complex. Then* X *has the homotopy type of the sum* $R_{ik} \vee \alpha S^{ik}$ *of* R_{ik} *with* $\alpha = (\chi(X) - 1)$ *copies of the* ik*-sphere* S^{ik}.

(b) *Let* $m = ik - 1$ $(i > 0)$ *and* Y *be any* $(\pi, ik - 1)$*-complex whose Euler characteristic* $\chi(Y) < 0$. *Then* Y *has the homotopy type of* $R_{ik-1} \vee \beta S^{ik-1}$ *with* $\beta = -\chi(Y)$.

(c) *The set of homotopy classes* $HT(\pi, ik - 1)_0$ *of* $(\pi, ik - 1)$*-complexes with Euler characteristic zero is isomorphic to the group*

$$HT(\pi, ik - 1)_0 \cong \ker \{\nu: \mathbf{Z}_n^* \to \widetilde{K}_0\, \mathbf{Z}\pi\} / \pm(\mathrm{Aut}_k \pi)^i.$$

Thus the tree of homotopy types $HT(\pi, m)$ *has the appearance of Figure* 1:

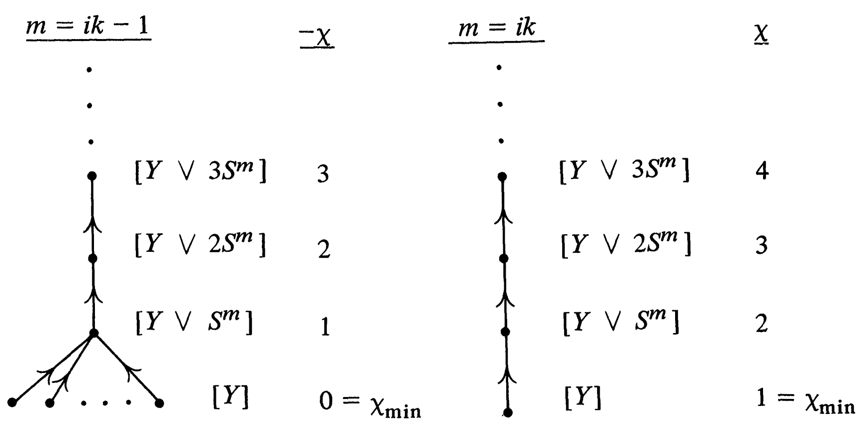

FIGURE 1

For example, if n is odd, the dihedral groups D_{2n} of order $2n$ satisfy the hypotheses of Theorem 1. For $\pi = \mathbf{Z}_n$, Theorem 1 gives a complete classification of the homotopy trees $HT(\mathbf{Z}_n, i)$ $(i \geqslant 2)$. The roots of $HT(\mathbf{Z}_n, \text{odd})$ are the homotopy classes of the standard lens spaces, and 1(c) suitably translated gives the classical homotopy classification of lens spaces [2, p. 96]. $HT(\mathbf{Z}_n, 2)$ was previously known (see [**1**] for n prime, [**4**] for arbitrary n).

The proof uses the theory of algebraic m-types [**1**], [**6**], [**11**], the module cancellation theory of H. Jacobinski [5], [8, p. 178], [3], the periodic resolution theory of R. Swan [7], and the following new application of the Wall obstruction.

DEFINITION. Let X be a connected CW complex of finite type (each skeleton $X^{(i)}$ is a finite complex, $i \geqslant 0$). If $H_m(\widetilde{X}, \widetilde{X}^{(m-1)})$ is a projective $\pi_1(X)$-module, then the *Swan-Wall class* $SW_m[X]$ is the class of $H_m(\widetilde{X}, \widetilde{X}^{(m-1)})$ in $\widetilde{K}_0 Z\pi_1(X)$ [10].

A connected CW complex X has the same *topological m-type* [11] as a connected complex Y if and only if there are maps

$$f\colon X^{(m+1)} \rightleftarrows Y^{(m+1)}\colon g$$

such that

$$g \circ f|_{X^{(m)}} \simeq (X^{(m)} \hookrightarrow X^{(m+1)}) \quad \text{and} \quad f \circ g_{\,Y^{(m)}} \simeq (Y^{(m)} \hookrightarrow Y^{(m+1)}).$$

The maps f and g are called m-homotopy equivalences. We say that $SW_m[X] \in \widetilde{K}_0 Z\pi_1(X)$ is an *invariant of the topological m-type of* X if, for any complex Y and m-equivalence $f\colon X^{(m+1)} \longrightarrow Y^{(m+1)}$, the homomorphism $\widetilde{K}_0 f_{1*}\colon \widetilde{K}_0 Z\pi_1(X) \longrightarrow \widetilde{K}_0 Z\pi_1(Y)$ induced by $f_{1*}\colon \pi_1(X) \longrightarrow \pi_1(Y)$ carries $SW_m[X] \longrightarrow SW_m[Y]$.

THEOREM 2. *If $\pi_1(X)$ is a finite group, then, if defined, the Swan-Wall class $SW_m[X]$ $(m \geqslant 2)$ is an invariant of the topological m-type of X.*

To each connected CW complex X having $\widetilde{X}(m-1)$-connected and $\pi_1(X) = \pi$, we associate its algebraic m-type $(\pi, \pi_m(X), k(X))$, where the cohomology class $k(X) \in H^{m+1}(\pi, \pi_m(X))$ is the obstruction invariant of [6]. An *abstract algebraic m-type* is a triple $\mathbf{T} = (\pi, \pi_m, k)$, where π is a multiplicative group, π_m a π-module, and k a class in $H^{m+1}(\pi, \pi_m)$. Two algebraic m-types $\mathbf{T} = (\pi, \pi_m, k)$ and $\mathbf{T}' = (\pi', \pi'_m, k')$ are *isomorphic* $(\mathbf{T} \cong \mathbf{T}')$ if there are isomorphisms $f\colon \pi \longrightarrow \pi'$, $f'\colon \pi_m \longrightarrow \pi'_m$, where f is a group homomorphism, f' is an f-homomorphism $(f'(x \cdot a) = f(x) \cdot f'(a)$ for $x \in \pi$, $a \in \pi_m)$, and $f'_*(k) = f^*(k')$ in the diagram

$$H^{m+1}(\pi, \pi_m) \xrightarrow{f'_*} H^{m+1}(\pi, (\pi'_m)_f) \xleftarrow{f^*} H^{m+1}(\pi', \pi'_m).$$

Here $(\pi'_m)_f$ is the π-module with action given by

$$x * a' = f(x) \cdot a' \; (x \in \pi, a' \in \pi'_m).$$

It is known from [6] that two complexes X, Y whose universal covers are $(m-1)$-connected have the same m-type if and only if $\mathbf{T}(X) \cong \mathbf{T}(Y)$. Thus two (π, m)-complexes X, Y have the *same homotopy type* if and only if $\mathbf{T}(X) \cong \mathbf{T}(Y)$. It is also known from [6] that every abstract m-type $\mathbf{T} = (\pi, \pi_m, k)$ can be realized by a connected $(m+1)$-dimensional complex Y in the sense that $\mathbf{T}(Y) \cong \mathbf{T}$. Theorem 2 allows one to decide whether an algebraic m-type is realizable as a (π, m)-complex or not.

THEOREM 3. *Let π be a finite group. Let $\mathbf{T} = (\pi, \pi_m, k)$ be an abstract m-type and suppose X is any $(m+1)$-dimensional finite connected complex such that $\mathbf{T}(X) \cong \mathbf{T}$. Then there is a (π, m)-complex Y such that $\mathbf{T}(Y) \cong \mathbf{T}$ if and only if $SW_m[X] = 0$, provided $m > 2$* [**10**, *Theorem* F].

We can define a Swan-Wall class $SW_m[\mathbf{T}]$ for an algebraic m-type, provided $\exists X$ having $\mathbf{T}(X) \cong \mathbf{T}$ and $SW_m[X] \in \widetilde{K}_0\mathbf{Z}\pi$. Consider the natural action of $\operatorname{Aut}\pi$ on $\widetilde{K}_0\mathbf{Z}\pi$ and let $A_0\mathbf{Z}\pi$ be the group of orbits under this action. By Theorem 2, $SW_m[\mathbf{T}]$ is a well-defined member of $A_0\mathbf{Z}\pi$. However, if π is finite periodic, in many cases the Swan-Wall class contains only a single element. For example, $[(p, N)]$ is fixed under the action of $\operatorname{Aut}\pi$.

SKETCH OF A PROOF OF 1(b). Let Y be a (π, m)-complex such that $-\chi(Y) > 0$. Because π is finite, the k-invariant $k(Y)$ is a generator of $H^{m+1}(\pi, \pi_m(Y)) \cong \mathbf{Z}_n$, where n is the order of π. We assign to the space R of maximal Euler characteristic the m-type $\mathbf{T}(R) = (\pi, \mathbf{Z}, 1 \in \mathbf{Z}_n)$. By [**11**, Theorem 14] there is a π-isomorphism $\mathbf{Z} \oplus (\mathbf{Z}\pi)^i \longrightarrow \pi_m(Y) \oplus (\mathbf{Z}\pi)^j$ $(i > j)$. If π is abelian or 8 does not divide n, then π satisfies the Eichler condition [8, p. 177], and hence there is an isomorphism $\mathbf{Z} \oplus (\mathbf{Z}\pi)^\beta \cong \pi_m(Y)$ $(\beta = -\chi(Y) > 0)$. Thus $\mathbf{T}(Y) \cong (\pi, \mathbf{Z} \oplus (\mathbf{Z}\pi)^\beta, p)$ for some $p \in \mathbf{Z}_n^*$. By Theorem 2, the Swan-Wall class of $\mathbf{T}_p = (\pi, \mathbf{Z}, p)$ is well defined (up to action by $\operatorname{Aut}\pi$) as a member of $\widetilde{K}_0\mathbf{Z}\pi$ and by [7], $SW_m[\mathbf{T}_p] = SW_m[\mathbf{T}(Y)] = \nu(p)$. It follows from Theorem 3 that $\nu(p) = 0$. Then Lemma 6.1 of [7] provides an isomorphism

$$\mathbf{T}(R \vee \beta S^m) \cong (\pi, \mathbf{Z} \oplus (\mathbf{Z}\pi)^\beta, 1)$$

$$\cong (\pi, \mathbf{Z} \oplus (\mathbf{Z}\pi)^\beta, p) \cong \mathbf{T}(Y);$$

hence $R \vee \beta S^m \simeq Y$. This completes the proof.

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