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CALCULUS OF VARIATIONS: PERTURBATIONS PRESERVING CONDITION (C)[1]

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I. **Introduction.** Let E be a C^∞ fiber bundle over a C^∞ finite-dimensional compact Riemannian manifold M, possibly with boundary. The generalized variational "Dirichlet problem" studies the critical points of a functional $J(u) = \int L(u)$, where L, the Lagrangian, is a differential operator from sections of E with prescribed boundary values to sections of the trivial line bundle, $\mathbf{R}_M$. We consider existence, and especially the question: if J satisfies the Palais-Smale condition (C) (which insures the existence of critical points if J is bounded below [3], [5]), then under what conditions on a perturbation V can one show that $J - V$ also satisfies condition (C). One part of this involves investigating the more classical question of finding conditions on L such that $\|u\| \longrightarrow \infty$ implies $|J(u)| \longrightarrow \infty$. This is an important ingredient in using monotonicity methods in proving existence theorems for nonlinear partial differential equations [1].

For a smooth vector bundle ξ over M, let $L^p_k(\xi)$ be the Sobolev space of sections whose covariant derivatives up through order k are in $L^p(\xi)$, with norm $\| \ \|_{p,k}$, while $L^p_k(\xi)_0$ is the subspace with "zero boundary values". It is easy to see that a functional J having the following two properties satisfies condition (C).

(a) J is *pseudo-proper* if $\|u\|_{p,k} \longrightarrow \infty$ implies $|J(u)| \longrightarrow \infty$.

(b) J is *coercive* if given any bounded sequence u_j such that $(DJ_{u_i} - DJ_{u_j})(u_i - u_j) \longrightarrow 0$, then u_j has an L^p_k convergent subsequence.

(In the literature property (a) is sometimes referred to as the coercive condition.)

Our results are of two sorts. First, assuming that J_0 is pseudo-proper

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(resp. coercive), we find conditions on a perturbation $\mathcal{V} = \int V$ such that $J = J_0 - \mathcal{V}$ is also pseudo-proper (resp. coercive). This is more than a stability question. It is not difficult to show that if $\mathcal{V}$ is "small" enough, then it preserves the two conditions. We ask, rather, how large $\mathcal{V}$ can be. Second, we investigate necessary conditions on L such that $J = \int L$ is pseudo-proper. There is a fine line between perturbation results and necessity results, so this delineation may occasionally seem arbitrary.

For ease in exposition, we will not attempt to give the most general version of any of our results.

II. **Perturbation theory**. We begin with an especially illuminating and useful special case. For $u \in L_1^2(M, \mathbf{R})_0$, let $J_0(u) = \int |\nabla u|^2 = \|u\|_{2,1}^2$. Let V: $M \times \mathbf{R} \longrightarrow \mathbf{R}$ be continuous, and $\lambda_1 > 0$ be the first eigenvalue of the Laplacian. Finally, let $J(u) = J_0(u) - \int V(x, u)$.

THEOREM 1$'$. (a) *Assume*

$$V(x, s) \leqslant \text{const} + Ks^2 \quad \textit{for all} \;\; s \in \mathbf{R}. \tag{1$'$}$$

If

$$K < \lambda_1, \tag{2$'$}$$

then J *is bounded below and pseudo-proper.*

(b) *If*

$$V(x, s) \geqslant \text{const} + \lambda_1 s^2, \tag{3$'$}$$

then J *is not pseudo-proper.*

Part (a) of Theorm 1$'$ says that if V grows at most quadratically, with a growth rate bounded by λ_1, then J will be pseudo-proper. This condition on V can be viewed as an asymptotic growth estimate; compare with [**6**, §8]. Unlimited growth in the "negative" direction is allowed. For example, $V(x, u) = -e^u + a(\sin u)u^2$ satisfies inequality (1$'$). Part (b) shows the growth restrictions in (a) are sharp.

To prove part (a), we observe that λ_1 is the best constant in the Sobolev inequality

$$\int u^2 \leqslant \frac{1}{\lambda_1}\int |\nabla u|^2, \qquad u \in L_1^2(M, \mathbf{R})_0.$$

For (b), we explicitly construct a sequence of functions u_j such that $|J(u_j)|$ is bounded, but $\|u_j\|_{2,1} \longrightarrow \infty$.

Theorem 1′ extends to more general functionals J_0 on $L^p_k(\xi)$ with prescribed boundary values. Here we give the case $J_0(u) = \|u\|^p_{p,k}$ on $L^p_k(\xi)_0$, where $\xi = \Omega \times \mathbf{R}$, for a bounded open domain $\Omega \subset \mathbf{R}^n$.

THEOREM 1. (a) *Assume that*

(1) $$V(x, D^\alpha u) \leqslant \text{const} + \Sigma\, a_\beta(x) D^{\beta_1}u \cdots D^{\beta_p}u,$$

where $|\alpha| \leqslant k - 1$, *and* $\beta = (\beta_1, \cdots, \beta_p)$ *is a p-tuple of multi-indices with* $|\beta_j| \leqslant k - 1$. *Then there are "best" constants* K_β *such that for all* $\gamma > K_\beta$,

$$\int a_\beta(x) D^{\beta_1}u \cdots D^{\beta_p}u \leqslant \gamma \|u\|^p_{p,k}.$$

(2) *Moreover, if* $\Sigma K_\beta < 1$, *then* $J_0 - \int V$ *is bounded below and pseudo-proper.*

(b) *If there is a differential operator* $P(x, D^\alpha u)$, *homogeneous in* u *of degree* $r > p$, *and* $\psi \in C^\infty_0(\Omega)$ *such that* $P(x, D^\alpha \psi) > 0$, *and*

(3) $$V(x, D^\alpha u) \geqslant \text{const} + P(x, D^\alpha u),$$

then $J_0 - \int V$ *is not pseudo-proper. In fact,* (3) *need only hold on some open set in* Ω.

We can give an explicit upper bound for each K_β in terms of constants in the Sobolev embedding theorems.

For our results on coercivity, we need the notion of the *weight* of a polynomial differential operator (one that locally is a polynomial in the derivatives of u, whose coefficients may depend on u, but not its derivatives). The weight of a monomial $a(x, u)D^{\beta_1}u \cdots D^{\beta_m}u$ is $|\beta_1| + \cdots + |\beta_m|$; the weight of a polynomial is the largest weight that occurs in its monomials. If $pk > n = \dim M$, we can give $L^p_k(E)$ the structure of a C^∞ infinite-dimensional Finsler manifold modeled on $L^p_k(\xi)$ [2]. We work on this manifold of maps.

Consider $J_0(u) = \int L(u)$, where L is a polynomial differential operator of order k and weight pk (the weight condition insures L is smooth from $L^p_k(E)$ to $L^1_0(\mathbf{R}_M)$). The perturbations will also be polynomial of weight at most pk, and will only depend on the $(k-1)$-jet of u. Under these hypotheses, we have the best possible result.

THEOREM 2. *Let* $J_0(u) = \int L(u)$ *as above, and* $V(u)$ *be a polynom-*

ial differential operator of order $k - 1$. *If* J_0 *is coercive on* $L^p_k(E)$, *and the weight of* V *is at most* pk, *then* $J = J_0 - \int V$ *is coercive on* $L^p_k(E)$.

The main ingredient of the proof is to show that if $\|u\|_{p,k}$ is bounded, then $(DV_{u_i} - DV_{u_j})(u_i - u_j) \longrightarrow 0$. If pk is much bigger than n, then V can have higher weight. For example, if $p > n$, then V can be any C^1 function of the $(k-1)$-jet of u.

III. **Necessary conditions.** We seek algebraic necessary conditions for polynomial Lagrangians to be pseudo-proper. To begin with, we note that for a strict polynomial Lagrangian (coefficients are polynomial in u) to give rise to a pseudo-proper functional on $L^p_k(\xi)$, p must be an even integer. Now we show that certain perturbations preserve pseudo-properness.

THEOREM 3. *Let* J_0: $L^p_k(\xi) \longrightarrow \mathbf{R}$ *be pseudo-proper, and homogeneous of degree* p. *If* $\mathcal{V}$: $L^p_k(\xi) \longrightarrow \mathbf{R}$ *consists of a finite sum of terms continuous on* $L^p_k(\xi)$ *and homogeneous of degree less than* p, *then* $J = J_0 - \mathcal{V}$ *is pseudo-proper.*

If L is a linear differential operator of order k, then one can show that $J(u) = \int |Lu|^p$ is pseudo-proper on $L^p_k(\xi)_0$ if and only if L is elliptic, with trivial kernel. Combined with Theorem 3, this implies that $\int (\Delta u)^4 \pm (u_{xx} - u_{yy})^3$ is pseudo-proper, despite the hyperbolicity of the $u_{xx} - u_{yy}$ perturbation.

There are simple counterexamples to the converse of Theorem 3. However, we can prove a partial converse, which essentially says that if $J = J_0 - \mathcal{V}$, as above, is pseudo-proper, then so is J_0, modulo a term of the form $\int |u|^p$.

The moral of Theorem 3 and its converse, is that the "principal term" of a polynomial Lagrangian in the determination of pseudo-properness, is the one of highest order and highest homogeneity. Our main result shows a positivity condition on the principal term is necessary for pseudo-properness.

Let $J(u) = \int \mathsf{L}(u)$, where L is a kth order strict polynomial differential operator from $\xi = \Omega \times \mathbf{R}$, $\Omega \subset \mathbf{R}^n$, to $\mathbf{R}_M$, homogeneous of degree p. We can write

$$\mathsf{L}(u) = P(x; D^\alpha u) + Q(x; D^\alpha u), \qquad |\alpha| \leqslant k, \tag{4}$$

where P and Q are polynomial differential operators with smooth coefficients in x, homogeneous of degree p; P involves only kth order derivatives of u, and Q has no term consisting of a product of p kth order derivatives.

THEOREM 4. *If $J(u)$ is as above, and pseudo-proper on $L_k^p(\xi)$, then P is "positive", in the sense that there is a constant $c > 0$ such that for all $x \in \overline{\Omega}$ and all $\zeta \in \mathbf{R}^n$,*

$$|\zeta|^{pk} \leqslant c|P(x; \zeta^\alpha)|. \tag{5}$$

For the case $p = 2$, condition (5) says the bilinear form associated with $P(x; D^\alpha u)$ is uniformly strongly elliptic. In this case, using Gårdings inequality, one can show that if L has the form (4), and (5) holds for all $x \in \overline{\Omega}$, then there is a constant $K > 0$ such that $J(u) + K\|u\|_{2,0}^2$ is pseudo-proper on $L_k^2(\xi)$.

In proving Theorems 3 and 4, we use the observation that if a functional J is pseudo-proper on $L_k^p(\xi)$, and homogeneous of degree $r > 0$, then there is a constant $C > 0$ such that $\|u\|_{p,k}^p \leqslant C|J(u)|^{p/r}$.

This inequality is reminiscent of the fundamental inequality for linear elliptic operators.

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