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ISOMETRIC MINIMAL IMMERSIONS OF $S^3(a)$ IN $S^N(1)$

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Introduction. We denote by $S^p(a)$ the sphere of radius **a** in the euclidean $(p+1)$-space E^{p+1}, with the induced metric. In [**1**], S. S. Chern asks the following question: "Let $S^3(a) \to S^7(1)$ be an isometric minimal immersion. Is it totally geodesic?". In this note we announce the following result.

THEOREM. *Let* $S^3(a) \subset E^4 \to S^N(1) \subset E^{N+1}$ *be an isometric minimal immersion which is not totally geodesic. Then* $N \geqslant 8$.

The class of isometric minimal immersions of $S^p(a) \to S^N(1)$ was qualitatively described by M. do Carmo and N. R. Wallach in [**3**]. For $p = 2$, each admissible **a** determines a unique element of such a class. The main result of [**3**] shows that for each $p \geqslant 3$ and each admissible $a \geqslant \sqrt{8}$, there exists a continuum of distinct such immersions. Our Theorem is an answer to a question of quantitative character. This constitutes part of our doctoral dissertation at IMPA. I want to thank my adviser Professor M. do Carmo for suggesting this problem and for helpful conversations.

Definitions and lemmas. Let $H = (\varphi_0, \cdots, \varphi_N)$: $S^3(a) \subset E^4 \to S^N(1) \subset E^{N+1}$ be an isometric minimal immersion. Then [**1**] the coordinate functions are spherical harmonics on $S^3(a)$, i.e., each φ_i $(0 \leqslant i \leqslant N)$ is the restriction to $S^3(a)$ of a homogeneous polynomial of degree s, with four indeterminates satisfying the condition

$$\sum_{k=1}^{4} \frac{\partial^2 \varphi_i}{\partial x_k^2} = 0. \tag{1}$$

Initially we set

$$\varphi_i = \sum_{\Sigma \alpha_i = s} a_{\alpha_1 \cdots \alpha_4} x_1^{\alpha_1} \cdots x_4^{\alpha_4}, \tag{2}$$

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and write

$$H = \sum_{\Sigma\alpha_i = s} A_{\alpha_1\cdots\alpha_4} x_1^{\alpha_1}\cdots x_4^{\alpha_4}, \tag{3}$$

where the vectors $A_{\alpha_1\cdots\alpha_4} = (a^0_{\alpha_1\cdots\alpha_4}, \cdots, a^N_{\alpha_1\cdots\alpha_4}) \in E^{N+1}$ are the column-vectors of the matrix in which the ith row is made up of the coefficients of φ_i $(0 \leqslant i \leqslant N)$. We denote by $V(H)$ the subspace of E^{N+1} generated by the vectors $A_{\alpha_1\cdots\alpha_4}$.

Identify the set

$$X^s = \left\{x_1^{\alpha_1}\cdots x_4^{\alpha_4};\ \sum \alpha_i = s,\ \alpha_i \geqslant 0,\ \text{integer}\right\}$$

with the tetrahedron

$$T^s = \left\{(\alpha_1, \cdots, \alpha_4, 0, \cdots, 0) \in E^{N+1};\ \sum \alpha_i = s,\ \alpha_i \geqslant 0,\ \text{integer}\right\}$$

by means of the correspondence

$$x_1^{\alpha_1}\cdots x_4^{\alpha_4} \longleftrightarrow (\alpha_1, \cdots, \alpha_4, 0, \cdots, 0).$$

It can be shown that the equivalence relation defined on T^s: $(\alpha_1, \cdots, \alpha_4, 0, \cdots, 0) \in T^s$ and $(\alpha'_1, \cdots, \alpha'_4, 0, \cdots, 0) \in T^s$ are equivalent iff for each $i = 1, \cdots, 4$, $\alpha_i - \alpha'_i \equiv 0 \pmod 2$ decomposes T^s in 8 equivalence classes C_i, $i = 1, \cdots, 8$, if $s > 2$. Let π: $T^s \to E^{N+1}$ be the map

$$\pi(\alpha_1, \cdots, \alpha_4, 0, \cdots, 0) = A_{\alpha_1\cdots\alpha_4},$$

and denote by $V_i(H)$ the subspace of E^{N+1} generated by $\pi(C_i)$, $i = 1, \cdots, 8$.

With the above notation we have

LEMMA A. *Let* $H = \Sigma\, A_{\alpha_1\cdots\alpha_4} x_1^{\alpha_1}\cdots x_4^{\alpha_4}$ *be an isometry of* $S^3(a)$ *in* $S^N(1)$ *where* H *is given by* (3). *Then* $V(H) = V_1(H) \oplus \cdots \oplus V_8(H)$, *if* $\dim V_i(H) = 1$, $i = 1, \cdots, 8$.

LEMMA B. *Let* $H = (\varphi_0, \cdots, \varphi_N)$: $S^3(a) \subset E^4 \to S^N(1) \subset E^{N+1}$ *be an isometric minimal immersion which is not totally geodesic. Then the dimension of* $V(H)$ *is* > 8.

The Theorem follows immediately from Lemmas A and B.

Sketches of proofs. First we establish some general facts. We indicate the inner product of A, $B \in E^{N+1}$ by AB (or A^2 if $A = B$).

Under the conditions of Lemma A, we have

$$H^2 = \sum_{\Sigma \alpha_i = s; \Sigma \beta_i = s} A_{\alpha_1 \cdots \alpha_4} A_{\beta_1 \cdots \beta_4} x_1^{\alpha_1 + \beta_1} \cdots x_4^{\alpha_4 + \beta_4}, \tag{4}$$

and

$$1 = a^{-2s} \sum_{\Sigma \alpha_i = s} \frac{s!}{\alpha_1! \cdots \alpha_4!} x_1^{2\alpha_1} \cdots x_4^{2\alpha_4}. \tag{5}$$

Under the conditions of Lemma B, we have that the coordinate functions φ_i, $i = 0, \cdots, N$, are spherical harmonics. This implies for each element in T^{s-2} a linear relation of the type

$$\begin{aligned} \alpha_1(\alpha_1 - 1)A_{\alpha_1\alpha_2\alpha_3\alpha_4} &+ (\alpha_2 + 2)(\alpha_2 + 1)A_{(\alpha_1-2)(\alpha_2+2)\alpha_3\alpha_4} \\ &+ (\alpha_3 + 2)(\alpha_3 + 1)A_{(\alpha_1-2)\alpha_2(\alpha_3+2)\alpha_4} \\ &+ (\alpha_4 + 2)(\alpha_4 + 1)A_{(\alpha_1-2)\alpha_2\alpha_3(\alpha_4+2)} = 0. \end{aligned} \tag{6}$$

Then using the fact that H preserves inner products, we compare the products of tangent vectors at convenient paths on $S^3(a)$ with those at the transformed paths on $H(S^3(a))$. This yields relations between the vectors $A_{\alpha_1 \cdots \alpha_4}$. We exhibit some typical relations.

$$\begin{aligned} A_{s000}^2 = a^{-2s}; \quad A_{(s-1)100}^2 = a^{2(1-s)}; \quad A_{s000}A_{(s-2)200} < 0; \\ A_{(s-2)110}^2 + 2A_{(s-1)100}A_{(s-3)120} = (s-1)a^{2(1-s)}; \\ 2A_{(s-2)200}^2 + 3A_{(s-1)100}A_{(s-3)300} = 6^{-1}s(s-1)(s+4)a^{-2s}; \\ 2A_{(s-2)200}A_{(s-2)020} + A_{(s-1)100}A_{(s-3)120} = -6^{-1}s^2(s-1)a^{-2s}. \end{aligned} \tag{7}$$

Similar relations are obtained for each permutation of 2 indices in $A_{\alpha_1\alpha_2\alpha_3\alpha_4}$.

Lemma A. From the definitions of $V(H)$ and $V_i(H)$, $i = 1, \cdots, 8$, we have $V(H) = V_1(H) + \cdots + V_8(H)$. The lemma follows immediately from the fact that if $A_{\alpha_1 \cdots \alpha_4} \in V_i(H)$ and $A_{\beta_1 \cdots \beta_4} \in V_j(H)$, with $i \neq j$, then $A_{\alpha_1 \cdots \alpha_4} A_{\beta_1 \cdots \beta_4} = 0$. In order to show this, we first observe that $\alpha_k + \beta_k$ is odd for some $k = 1, \cdots, 4$ (this follows from the definition of $V_i(H)$). Next we use (4), (5) and $H^2 = 1$, and observe that the terms in (5) with odd exponents are zero. This proves our claim.

Lemma B. This is the crucial point of the proof. The proof of Lemma B is reduced to obtaining estimates for the dimensions of $V_i(H)$, $i = 1, \cdots, 8$. Such estimates follow from the study of the relations between the vectors $A_{\alpha_1 \cdots \alpha_4}$, obtained from (6) and (7). We first show that if s is odd (even), then four (seven) of the subspaces $V_i(H)$ necessarily have dimension $\geqslant 1$. Next we examine the hypothesis of nullity of some $V_i(H)$ and conclude that in all cases the sum of the dimensions of the $V_i(H)$, $i = 1, \cdots, 8$, is greater than 8. The final and more delicate case occurs when we assume that the dimension of each $V_i(H)$, $i = 1, \cdots, 8$, is equal to 1. This assumption forces H to be totally geodesic, hence, a contradiction.

REFERENCES

1. S. S. Chern, *Brief survey of minimal submanifolds.* I, Mathematischen Forschungsinstitut, Oberwolfach, Band 4, 1971.

2. M. P. do Carmo, *Brief survey of minimal submanifolds.* II, Mathematischen Forschungsinstitut, Oberwolfach, Band 4, 1971.

3. M. P. do Carmo and N. R. Wallach, *Minimal immersions of spheres into spheres,* Ann. of Math. (2) 93 (1971), 43–62. MR 43 #4048.

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