BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 6, November 1974

# ON SPACES OF RIEMANN SURFACES WITH NODES[1]

BY LIPMAN BERS

Communicated by Samuel Eilenberg, May 13, 1974

This is a summary of results, to be published in full elsewhere, which strengthen and refine the statements made in a previous announcement [1].

A compact Riemann surface with nodes of (arithmetic) genus $p > 1$ is a connected complex space $S$, on which there are $k = k(S) \geqslant 0$ points $P_1, \cdots, P_k$, called *nodes*, such that (i) every node $P_j$ has a neighborhood isomorphic to the analytic set $\{z_1 z_2 = 0, |z_1| < 1, |z_2| < 1\}$, with $P_j$ corresponding to $(0, 0)$; (ii) the set $S \backslash \{P_1, \cdots, P_k\}$ has $r \geqslant 1$ components $\Sigma_1, \cdots, \Sigma_r$, called *parts* of $S$, each $\Sigma_i$ is a Riemann surface of some genus $p_i$, compact except for $n_i$ punctures, with $3p_i - 3 + n_i \geqslant 0$, and $n_1 + \cdots + n_r = 2k$; and (iii) we have

$$p = (p_1 - 1) + \cdots + (p_r - 1) + k + 1.$$

Condition (ii) implies that every part carries a *Poincaré metric*, and condition (iii) is equivalent to the requirement that the total Poincaré area of $S$ be $4\pi(p - 1)$.

From now on $p$ is kept fixed and the letter $S$, with or without subscripts or superscripts, always denotes a surface with properties (i)–(iii). If $k(S) = 0$, $S$ is called nonsingular; if $k(S) = 3p - 3$, $S$ is called *terminal.*

A continuous surjection $f: S' \longrightarrow S$ is called a *deformation* if for every node $P \in S$, $f^{-1}(P)$ is either a node or a Jordan curve avoiding all nodes and, for every part $\Sigma$ of $S$, $f^{-1}|\Sigma$ is an orientation preserving homeomorphism. Two deformations, $f: S' \longrightarrow S$ and $g: S'' \longrightarrow S$ are called *equivalent* if there are homeomorphisms $\varphi: S' \longrightarrow S''$ and $\psi: S \longrightarrow S$, homotopic to an isomorphism and to the identity, respectively, such that $g \circ \varphi = \psi \circ f$. The *deformation space* $D(S)$ consists of all equivalence classes $[f]$ of deformations onto $S$. To every node $P \in S$ belongs a *distinguished subset* consisting

---

*AMS (MOS) subject classifications* (1970). Primary 32G15; Secondary 30A46, 30A58, 14H15.

[1]Work partially supported by the NSF.

Copyright © 1974, American Mathematical Society

of all $[f] \in D(S)$ with $f^{-1}(P)$ a node of $f^{-1}(S)$.

We define a Hausdorff topology on $D(S)$ as follows. If $c$ is a closed curve on a part of $S$, denote by $|c|$ the length of the unique geodesic freely homotopic to $S$. Let $C$ be a finite set of closed curves on parts of $S$, $\epsilon$ a positive number, and $\omega: S' \longrightarrow S$ a deformation. We say that $\omega$ is $(C, \epsilon)$ *small* if for every Jordan curve $c'$ on a part of $S'$ such that $\omega(c')$ is a node, $|c'| < \epsilon$, and for every $c \in C, ||\omega^{-1}(c)| - |c|| < \epsilon$. A set $A \subset D(S)$ is called *open* if, for every $[f] \in A$, there is a finite set $C$ of closed curves on parts of $f^{-1}(S)$, and a number $\epsilon > 0$, such that whenever $\omega: S' \longrightarrow f^{-1}(S)$ is $(C, \epsilon)$ small, $[f \circ \omega] \in A$.

THEOREM 1. *$D(S)$ is a cell. There is an (essentially canonical) homeomorphism of $D(S)$ onto $\mathbf{C}^{3p-3}$ which takes each distinguished subset onto a coordinate hyperplane.*

A deformation $h: S \longrightarrow S_0$ induces a mapping $h_*: D(S) \longrightarrow D(S_0)$, called an *allowable mapping,* which takes each $[f] \in D(S)$ into $[h \circ f]$.

THEOREM 2. *Let $S$ and $S_0$ have the same genus, and let $k(S_0) = k(S) + l$. If $l = 0$, an allowable mapping $D(S) \longrightarrow D(S_0)$ is a homeomorphic bijection. If $l > 0$, an allowable mapping $D(S) \longrightarrow D(S_0)$ is a universal covering of the complement of $l$ distinguished subsets.*

The proofs of Theorems 1 and 2 use the so-called Fenchel-Nielsen coordinates (cf. [**1**, p. 51]). An inequality for Fenchel-Nielsen coordinates stated in [**1**] as Theorem XV (and previously conjectured by Mumford) implies

THEOREM 3. *Let $S_1, \cdots, S_m$ be all not isomorphic terminal surfaces of genus $p$. There are compact sets $K_j \subset D(S_j), j = 1, \cdots, m$, such that every $S$ is of the form $S = f^{-1}(S_j)$, $[f] \in K_j$, for some $j$.*

If $S$ is nonsingular, $D(S)$ can be identified with the Teichmüller space $T_p$ of closed Riemann surfaces of genus $p$. For *every* $S$, each point in $D(S)$, not belonging to a distinguished subset, has a neighborhood which can be naturally identified with a neighborhood in $T_p$. Thus an open dense set in $D(S)$ is a complex manifold. It follows that $D(S)$ has the structure of a ringed space.

THEOREM 4. *$D(S)$ is a complex manifold which can be realized as a bounded domain in $\mathbf{C}^{3p-3}$. The distinguished subsets of $D(S)$ are nonsingular analytic hypersurfaces which meet transversally.*

The proof utilizes the Kleinian groups constructed in [**1**, pp. 46–47]. The spaces $X_\alpha(S)$ used there are finite ramified coverings of $D(S)$. The following statement is almost obvious.

THEOREM 5. *Allowable mappings are holomorphic.*

Let $\Gamma(S)$ be the group of allowable self-mappings of $D(S)$ induced by all topological orientation preserving self-mappings of $S$, and let $\Gamma_0(S)$ be the subgroup induced by the automorphisms (conformal self-mappings) of $S$. Note that if $\gamma([f]) = [g]$ for some $\gamma \in \Gamma(S)$, then $f^{-1}(S)$ is isomorphic to $g^{-1}(S)$. The converse statement is, in general, false.

THEOREM 6. *The group* $\Gamma(S)$ *is discrete, the subgroup* $\Gamma_0(S)$ *is finite and is the stabilizer of* $[\mathrm{id}] \in D(S)$ *in* $\Gamma(S)$.

Let $M_p$ denote the *moduli space* (Riemann space) for genus $p$, that is, the set of all isomorphism classes $[S]$ of Riemann surfaces with nodes, of genus $p$. We define a Hausdorff topology in $M_p$ by calling a set $B \subset M_p$ open if, for every $[S] \in B$, there is a finite set $C$ of closed curves on parts of $S$, and an $\epsilon > 0$, such that $[S'] \in B$ whenever there is a $(C, \epsilon)$ small deformation $S' \longrightarrow S$. The moduli space of nonsingular Riemann surfaces of genus $p$ is known to be a complex space, and is an open dense subset of $M_p$. Hence $M_p$ has the structure of a ringed space.

There is a canonical mapping $D(S) \longrightarrow M_p$ which sends $[f] \in D(S)$ into $[f^{-1}(S)]$.

THEOREM 7. *The canonical mapping* $D(S) \longrightarrow M_p$ *is holomorphic. Furthermore,* $[\mathrm{id}] \in D(S)$ *has a neighborhood* $N$, *stable under* $\Gamma_0(S)$, *such that* $N/\Gamma_0(S)$ *is isomorphic to a neighborhood of* $[S]$ *in* $M_p$.

Theorems 3 and 7 imply the known (cf. [**2**])

COROLLARY (MAYER-MUMFORD). $M_p$ *is a compact normal complex space* (*and a* $V$-*manifold*).

A *regular* $q$-*differential* on $S$ is defined by assigning a holomorphic form $F_\Sigma$ of type $(q, 0)$ to each part $\Sigma$ of $S$; the $F_\Sigma$ should be either regular at the punctures, or have there poles of order not exceeding $q$, the "residues" at two punctures joined in a node being equal (if $q$ is even) or opposite (if $q$ is odd). The number $\delta(p, q)$ of linearly independent regular $q$-differentials is $p$ if $q = 1$, $(2q - 1)(p - 1)$ if $q > 1$. If we choose $\delta = \delta(p, q)$ linearly independent $q$-differentials, their "values" at every point of

$S$, including a node, are the homogeneous coordinates of a point in $\mathbf{P}_{\delta-1}$. In this way one obtains a holomorphic mapping $S \longrightarrow \mathbf{P}_{\delta-1}$, the so-called *q-canonical mapping.* This is an embedding for $q > 2$ and, in some cases, also for $q = 2$ and $q = 1$.

THEOREM 8. *For every $S$ and every $q \geqslant 1$, there is an analytic hypersurface $\sigma \subset D(S)$, with* $[\mathrm{id}] \notin \sigma$, *and a holomorphic mapping $\Phi$ of $D(S)\backslash\sigma$ into the Chow variety of curves of degree* $2q(p-1)$ *in $\mathbf{P}_{\delta(p,q)-1}$ such that, for* $[f] \in D(S)\backslash\sigma$, $\Phi([f])$ *is the Chow point of a q-canonical image of $f^{-1}(S)$.*

The proof uses the Poincaré series described in [1, pp. 48–49]. If $S$ is nonsingular, one knows, from other considerations, that the result is true with $\sigma = \varnothing$. For singular $S$, I could thus far obtain that $\sigma = \varnothing$ only for $q = 1$.

## REFERENCES

1. L. Bers, *Spaces of degenerating Riemann surfaces*, Discontinuous Groups and Riemann Surfaces, Ann. of Math. Studies, no. 79, Princeton Univ. Press, Princeton, N.J., 1974, pp. 43–55.

2. D. Mumford, *The structure of the moduli spaces of curves and Abelian varieties*, Proc. Internat. Congress Math. (Nice, 1970), Gauthier-Villars, Paris, 1971, pp. 457–465.

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY, NEW YORK, NEW YORK 10027