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A CRITERION FOR THE EXISTENCE OF BIHARMONIC GREEN'S FUNCTIONS

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The harmonic Green's function was originally introduced as the electrostatic potential of a point charge in a grounded system. Its characterization by the fundamental singularity and vanishing boundary values permitted its generalization to regular subregions Ω of an abstract Riemann surface or Riemannian manifold R. The Green's function g on R was then defined as the directed limit, if it exists, of the Green's function g_Ω on Ω as $\{\Omega\}$ exhausts R. The distinction of Riemann surfaces and Riemannian manifolds into hyperbolic and parabolic types according as g does or does not exist is still a cornerstone of the harmonic classification theory.

The biharmonic Green's function γ also has an important physical meaning: it is the deflection of a thin elastic plate under a point load. However, in sharp contrast with the harmonic case, nothing seems to be known about its existence on noncompact spaces. The purpose of the present paper is to initiate research on this fundamental problem of biharmonic classification theory.

Biharmonicity being not meaningful on abstract Riemann surfaces, our aim is to generalize the definition of the biharmonic Green's function to Riemannian manifolds R and to explore its existence on them. On a regular subregion Ω of R, there exist two biharmonic Green's functions, denoted by β and γ, with a biharmonic fundamental singularity, and with boundary data $\beta = \partial\beta/\partial n = 0$ and $\gamma = \Delta\gamma = 0$. For dimension 2, both functions give the deflection under a point load of a thin plate which is clamped or simply supported at the edges, respectively. Our present investigation deals exclusively with γ. The corresponding function γ_Ω on Ω increases with Ω, and we set $\gamma = \lim_{\Omega\to R}\gamma_\Omega$ on R.

We first study the existence of γ on the Euclidean N-space R^N. The result turns out to be quite fascinating: γ exists if and only if $N > 4$. By

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way of preparation we recall the peculiar behavior of the biharmonic fundamental singularity at the origin: $r^2 \log r$ for $N = 2$, r for $N = 3$, $\log r$ for $N = 4$, and r^{4-N} for $N > 4$.

For an arbitrary Riemannian manifold we deduce our main result: If $\bar{R}_0 \subset \Omega \subset R$, and ω_Ω is harmonic on $\Omega - \bar{R}_0$ with boundary values 1 on ∂R_0, 0 on $\partial\Omega$, and we denote by $\omega = \lim_{\Omega \to R} \omega_\Omega$ the harmonic measure of ∂R_0 on $R - R_0$, *then* γ *exists on* R *if and only if* $\omega \in L^2(R - R_0)$. An essential step of the reasoning is the proof of the independence of the existence of γ on the choice of the fundamental singularity. This property allows us to introduce the class O_Γ of Riemannian manifolds which do not carry γ, in analogy with the class O_G of parabolic manifolds.

As an illustration of our criterion we generalize the above result that $R^N \in O_\Gamma$ if and only if $N \leqslant 4$. We ask whether one could induce γ to exist even for these low dimensions by replacing the Euclidean metric $ds = |dx|$ by $ds = (1 + r^2)^\alpha |dx|$, with the constant α sufficiently large. The answer is intriguing: the resulting space is in O_Γ for $N \leqslant 4$ regardless of what α is chosen, whereas for $N > 4$, γ continues to exist if and only if $\alpha > -1/2$.

The usefulness of the criterion $\omega \in L^2$ for $R \notin O_\Gamma$ lies in the fact that it also applies if there is no way of obtaining an expression for the approximating Green's functions γ_Ω, and even if nothing is known about the metric off an arbitarily small neighborhood of the ideal boundary of the Riemannian manifold.

The proofs will appear in [1].

Relations of O_Γ to other biharmonic null classes will be discussed in another context.

REFERENCES

1. Leo Sario, *A criterion for the existence of biharmonic Green's functions* (to appear).

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