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ASYMPTOTIC NONUNIQUENESS OF THE NAVIER-STOKES EQUATIONS IN KINETIC THEORY[1]

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We consider the linearized Boltzmann equation

$$\partial p/\partial t + \xi \cdot \operatorname{grad} p = Qp/\epsilon, \tag{1}$$

whose solution $p = p_\epsilon(t, x, \xi)$, $t > 0, x \in R^3, \xi \in R^3, \epsilon > 0$. Q is the linearized collision operator corresponding to a spherically symmetric hard potential, and ϵ is a parameter which represents the mean free path.

In a series of basic papers, Grad [6], [7], [8] studied the existence and asymptotic behavior of the solution of the initial value problem for (1), where the initial data $p_\epsilon(0^+, x, \xi) = f(x, \xi)$ satisfies mild growth and smoothness conditions. Grad's method begins with the decomposition

$$Q = -\nu + K, \tag{2}$$

where ν is the operator of multiplication by the collision frequency $\nu(\xi)$, a strictly positive function of $|\xi|$, and K is a compact operator on the Hilbert space H_0 of functions $f(\xi)$ which satisfy

$$\langle f, f\rangle \equiv \left(\frac{1}{\sqrt{2\pi}}\right)^3 \int |f(\xi)|^2 \exp(-|\xi|^2/2)\, d\xi < \infty.$$

Using (2), Grad wrote (1) as an integral equation and then derived *a priori* estimates for the solution in the Hilbert space

$$H \equiv L^2(R^6, (1/\sqrt{2\pi})^3 \exp(-|\xi|^2/2)\, dx\, d\xi).$$

Grad also related the asymptotic behavior of p_ϵ to the solutions of the linear Euler and Navier-Stokes equations. Given $f \in H$, define

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$$f_0(x) = \langle f(x, \cdot), 1\rangle;$$
$$f_i(x) = \langle f(x, \cdot), \xi_i\rangle, \qquad i = 1, 2, 3;$$
$$f_4(x) = \langle f(x, \cdot), (|\xi|^2 - 3)/\sqrt{6}\rangle,$$

where $\langle \cdot , \cdot \rangle$ denotes the inner product on H_0. The Navier-Stokes equations are written

$$\text{(3)} \quad \begin{aligned} &\partial n_0/\partial t + \operatorname{div} \mathbf{n} = 0, \\ &\partial \mathbf{n}/\partial t + \operatorname{grad} n_0 + \sqrt{2/3}\operatorname{grad} n_4 = \epsilon\eta[\Delta \mathbf{n} + (1/3)\operatorname{grad}\operatorname{div}\mathbf{n}], \\ &\partial n_4/\partial t + \sqrt{2/3}\operatorname{div}\mathbf{n} = \epsilon\lambda\Delta n_4, \\ &n_i(0^+, \cdot) = f_i. \end{aligned}$$

In (3), $\epsilon > 0$, $n_i = n_i^\epsilon(t, x)$ $(i = 0, \cdots, 4)$, $\mathbf{n} = (n_1, n_2, n_3)$, and $\eta > 0$ and $\lambda > 0$ are physical constants. The Euler equations are obtained from (3) by putting $\epsilon = 0$. Setting

$$p_\epsilon = T_\epsilon(t)f,$$
$$N_\epsilon(t)f = n_0^\epsilon + \sum_{i=1}^{3} n_i^\epsilon \xi_i + n_4^\epsilon \frac{|\xi|^2 - 3}{\sqrt{6}},$$
$$E(t)f = N_0(t)f,$$

Grad proved the following asymptotic results:

$$\text{(4)} \qquad T_\epsilon(t)f = E(t)f + O(\epsilon), \qquad (\epsilon \downarrow 0)$$
$$\text{(5)} \qquad T_\epsilon(t/\epsilon)f = N_\epsilon(t/\epsilon)f + O(\epsilon).$$

In physical terms, (4) describes the nonviscous fluid approximation at a fixed time $t > 0$; (5) describes the viscous effects when $t \to \infty$. Our aim is to show that (5) is only one of a large variety of possible refinements of (4). This is accomplished by the following two results.

BOLTZMANN LIMIT THEOREM. *Let $f(x, \xi)$ be sufficiently regular. Then*

$$\text{(6)} \qquad E(-t/\epsilon)T_\epsilon(t/\epsilon)f = \bar{N}(t)f + O(\epsilon) \qquad (\epsilon \downarrow 0),$$

where $\bar{N}(t)$ is a contraction semigroup on H whose generator is given by the differential equations

$$
(7)\quad
\begin{aligned}
\frac{\partial n_0}{\partial t} &= \left(\frac{9}{25}\lambda + \frac{2}{5}\eta\right)\Delta n_0 + \sqrt{\frac{2}{3}}\left(-\frac{6}{25}\lambda + \frac{2}{5}\eta\right)\Delta n_4,\\
\frac{\partial \mathbf{n}}{\partial t} &= \eta\Delta \mathbf{n} + \left(\frac{\lambda}{5} - \frac{\eta}{3}\right)\text{grad div } \mathbf{n},\\
\frac{\partial n_4}{\partial t} &= \sqrt{\frac{2}{3}}\left(-\frac{6}{25}\lambda + \frac{2}{5}\eta\right)\Delta n_0 + \left(\frac{11}{25}\lambda + \frac{4}{15}\eta\right)\Delta n_4,
\end{aligned}
$$

$$n_i(0^+, x) = f_i(x);$$

i. e.,

$$\bar{N}(t)f = n_0 + \sum_1^3 n_i\xi_i + n_4\frac{|\xi|^2 - 3}{\sqrt{6}}.$$

The semigroup $\{\bar{N}(t), t \geq 0\}$ *commutes with the Euler semigroup* $\{E(t), t \geq 0\}$.

In order to make connection with (5) we also need the following.

NAVIER-STOKES LIMIT THEOREM. *Let* $f(x, \xi)$ *be sufficiently regular. Then*

$$(8)\qquad E(-t/\epsilon)N_\epsilon(t/\epsilon)f = \bar{N}(t)f + O(\epsilon) \qquad (\epsilon \downarrow 0).$$

The proof of (8) proceeds by means of Fourier transformation from the following purely algebraic result, of independent interest.

MATRIX LIMIT THEOREM. *Let* A, B *be real, symmetric* $m \times m$ *matrices and assume that* B *is negative semidefinite. Then*

$$\exp(-itA/\epsilon)\exp(t(iA + \epsilon B)/\epsilon) = \exp(t\pi_A B) + O(\epsilon) \qquad (\epsilon \downarrow 0),$$

where $\pi_A B$ *is the orthogonal projection, in the space of* $m \times m$ *matrices, of* B *onto the linear subspace of matrices which commute with* A.

In particular, we show that $\bar{N}(t)$ is obtained by a projection, in the space of operators, of $N_\epsilon(t)$ upon the set of operators which commute with $\{E(t), t \geq 0\}$.

Using (6), we have

$$(9)\qquad T_\epsilon(t/\epsilon)f = E(t/\epsilon)\bar{N}(t)f + O(\epsilon) \qquad (\epsilon \downarrow 0).$$

This is the simplest of an infinite number of alternatives to (5). Indeed, if $\tilde{N}(t)$ is any operator whose projection is $\bar{N}(t)$, then we may substitute $\tilde{N}(t)$ for $\bar{N}(t)$ in (9).

The proof of (6) depends on a careful spectral analysis of the operator $Q - i(\gamma \cdot \xi)$, where $\gamma \in R^3$ is a parameter. We prove the existence and differentiability, for $|\gamma|$ sufficiently small, of the hydrodynamical eigenvalues and eigenfunctions $\{\alpha^{(j)}(\gamma), e^{(j)}(\gamma); j = 1, \cdots, 5\}$ which satisfy $\alpha^{(j)}(0) = 0$, $e^{(j)}(0) \in \operatorname{span}\{1, \xi_1, \xi_2, \xi_3, |\xi|^2\}$. We then prove a contour integral representation

$$\exp[t(Q - i(\gamma \cdot \xi))]f = \sum_{j=1}^{5} \exp(t\alpha^{(j)})(\gamma)\langle f, e^{(j)}(-\gamma)\rangle e^{(j)}(\gamma) + \frac{1}{2\pi i}\int_C e^{t\alpha} R(\alpha, \gamma)\frac{(Q - i(\gamma \cdot \xi))^2}{\alpha^2} f\, d\alpha, \tag{10}$$

where C is a vertical contour in the half plane $\operatorname{Re}\alpha < 0$ and $R(\alpha, \gamma) \equiv (Q - i(\gamma \cdot \xi) - \alpha)^{-1}$. The first term of (10) corresponds to the *Hilbert solution* and gives the connection with hydrodynamics. The second term is negligible in the hydrodynamic limit. In case $\nu(\xi) \sim |\xi|^\alpha$ as $|\xi| \to \infty$ $(\alpha > 0)$, the contour integral may be replaced by $\int_C e^{t\alpha} R(\alpha, \gamma) f\, d\alpha$, where the contour C is such that $\operatorname{Re}\alpha \to -\infty$ when $\operatorname{Im}\alpha \to \pm\infty$. The existence of the eigenvalues $\alpha^{(j)}(\gamma)$ follows by applying the implicit function theorem to the exact hydrodynamical dispersion laws. Previously, exact dispersion laws were obtained [11] only for hard sphere potentials, i.e., $\nu(\xi) \sim |\xi|$ as $|\xi| \to \infty$. In this case, the $\alpha^{(j)}(\gamma)$ are analytic functions and can also be obtained from Rellich's perturbation theorem [9], [10]. In case $\nu(\xi) \sim |\xi|^\alpha$ as $|\xi| \to \infty$, $0 \leqslant \alpha < 1$, the $\alpha^{(j)}(\gamma)$ will not be analytic around $\gamma = 0$. Nevertheless, we obtain an asymptotic development

$$\alpha^{(j)}(\gamma) \sim \sum_{n=1}^{\infty} \alpha_n^{(j)} |\gamma|^n \qquad (1 \leqslant j \leqslant 5),$$

where $\alpha_1^{(j)}$ is imaginary and $\alpha_2^{(j)} < 0$. These constants can be computed by formal perturbation theory. They correspond to the adiabatic sound speed and absorption coefficients for low frequency sound waves [5].

The results (6) and (8) extend known results on finite-state velocity models in one dimension [1], [2] to the full three-dimensional linearized Boltzmann equation. These theorems are valid in any number of dimensions. Their proofs and related matters will appear in full detail in [3], [4].

REFERENCES

1. R. Ellis and M. Pinsky, *Limit theorems for model Boltzmann equations with several conserved quantities,* Indiana Univ. Math. J. 23 (1973), 287–307. MR 47 #8058.

2. ———, *Asymptotic equivalence of the linear Navier-Stokes and heat equations in one dimension*, J. Differential Equations (to appear).

3. R. Ellis and M. Pinsky, *Projection of the Navier-Stokes equations upon the Euler*

equations, J. Math. Pures Appl. (to appear).

4. ———, *The first and second fluid approximations to the Boltzmann equation*, J. Math. Pures Appl. (to appear).

5. J. D. Foch and G. W. Ford, *The dispersion of sound in monoatomic cases*, Studies in Statistical Mechanics, vol. 5, North-Holland, Amsterdam, 1970.

6. H. Grad, *Asymptotic theory of the Boltzmann equation.* II, Rarefied Gas Dynamics (Proc. Third Internat. Sympos., Palais de l'UNESCO, Paris, 1962), vol. 1, Academic Press, New York, 1963, pp. 26–59. MR 27 #6577.

7. ———, *Asymptotic equivalence of the Navier-Stokes and non-linear Boltzmann equations*, Proc. Sympos. Appl. Math., vol. 17, Amer. Math. Soc., Providence, R.I., 1965, pp. 154–183. MR 32 #1979.

8. ———, *Solution of the Boltzmann equation in an unbounded domain*, Comm. Pure Appl. Math. 18 (1965), 345–354. MR 32 #8913.

9. T. Kato, *Perturbation theory for linear operators*, Die Grundlehren der math. Wissenschaften, Band 132, Springer-Verlag, New York, 1966. MR 34 #3324.

10. J. A. McLennan, *Convergence of the Chapman-Enskog expansion for the linearized Boltzmann equation*, Phys. Fluids 8 (1965), 1580–1584. MR 33 #6948.

11. B. Nicolaenko, *Dispersion laws for plane wave propagation*, The Boltzmann Equation (ed. by F. A. Grünbaum), Courant Institute of Mathematical Sciences, New York, 1971, pp. 125–173.

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