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OPEN AND UNIFORMLY OPEN RELATIONS

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ABSTRACT. A general open mapping theorem is proved when the domain is an Efremovič proximity space. This is then used to prove several results on relations which are generalizations of results due to Kelley, Pettis and Weston. Applications to functional analysis are given.

1. **Introduction.** In a recent topology conference at Charlotte, Professor B. J. Pettis posed several problems concerning open and uniformly open relations. This paper is a brief announcement of the results of our investigation to answer some of these questions; details with proofs will appear elsewhere. Most of the terms used are well known and will be found in Kelley [3]. If (X, δ) is a proximity space, Y a topological space and $R \subset X \times Y$ is a relation, then R is *weakly open* iff for each $y \in Y, A \subset X, y \in R[A]^-$ implies $R^{-1}[y]\ \delta\ A$ (Poljakov [7]). If R is injective and open then R is weakly open. Also if (X, d) is a metric space (with the induced metric proximity), Y is a Morita uniform space and R is uniformly open, then R is weakly open.

2. **Main results.**

2.1 THEOREM. *If (X, δ) is an Efremovič proximity space, Y a topological space, $R \subset X \times Y$ a weakly open and nearly open relation, then R is open.*

2.2 THEOREM. *If (X, d) is a metric space, (Y, Ω) a Morita uniform space, $R \subset X \times Y$ is weakly open and uniformly nearly open if and only if R is uniformly open.*

2.3 THEOREM. *If (X, δ) is an Efremovič proximity space, Y a topological space, $R \subset X \times Y$ an injective relation, then R is open if and only if R is weakly open and nearly open.*

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3. Generalizations of Weston and Kelley theorems.

3.1 THEOREM (CF. WESTON [8, THEOREM 8]). *Let* (X, d) *be a complete metric space,* Y *a Hausdorff space,* $R \subset X \times Y$ *a point-compact, USC nearly open bijective relation. Then* R *is open.*

3.2 THEOREM (CF. KELLEY [3, THEOREM 6.36]). *Let* (X, d) *be a complete metric space,* (Y, Ω) *a Morita uniform space, and let* $R \subset X \times Y$ *have a closed graph. If* R *is uniformly nearly open, then* R *is uniformly open.*

If in (3.1) we replace Y by a T_4-space and R by a point-closed relation, the result remains true. Theorem 3.1 and the above modification give analogous dual results concerning the implication "nearly continuous implies continuous". Theorem (3.2) yields an improvement of a result of Brown [1] to closed graphs.

4. Recent results of Pettis.

Here we generalize recent results of Pettis [6].

4.1 THEOREM. *Suppose* Y *is Hausdorff,* $R \subset X \times Y$ *is a bijective nearly open relation with a metrically complete graph. Then* R *is open.*

4.2 THEOREM. *If* X *and* Y *are metrically complete,* $R \subset X \times Y$ *is a bijective nearly open relation with a closed graph, then* R *is open.*

Each of (4.1) and (4.2) includes two results of Pettis, one concerning open mappings and another concerning continuous mappings. We give a sample of the latter.

4.3 THEOREM (PETTIS [6]). *If* X *is Hausdorff and* $f: X \longrightarrow Y$ *is a nearly continuous function with a metrically complete graph, then* f *is continuous and* X *is metrically complete.*

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