BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 6, November 1974

# SUBCOMPLEXES OF POINCARÉ COMPLEXES

BY J. P. E. HODGSON

Communicated by Glen Bredon, March 12, 1974

1. **Introduction.** This paper is concerned with the following situation. Suppose we are given a simplicial complex $X$ which is a Poincaré duality space for some (twisted) coefficients, and that $K$ is a subcomplex of $X$ which is itself a Poincaré duality space; when can we deduce that the inclusion of $K$ in $X$ is homotopic to a Poincaré embedding?

There are two interesting cases of the above situation, the first is given by a theorem of Bredon [1], which states that if $Z_p$ acts on a simplicial complex which is a $Z_p$-Poincaré duality space, then each component of the fixed point set is a $Z_p$-Poincaré duality space. The other case is that of the product of a Poincaré duality space with itself, triangulated so that the diagonal is a subcomplex. We shall give a condition sufficient to obtain an embedding in these two cases, subject to certain dimensional restrictions.

2. **Normally embedded subcomplexes.** For the purpose of simplicity of exposition, we shall restrict ourselves to the simply-connected case. We shall begin by discussing the case of integer coefficients, and will describe the modifications required for the $Z_p$ case when we discuss the Bredon theorem. We assume familiarity with the basic definitions of Poincaré complex, Poincaré embedding (they can be found for instance in Levitt [3]), and with the concept of a normal space and normal pair as defined by Quinn [4].

It is known [4] that the mapping cylinder of a normal map is a normal pair. Furthermore a Poincaré complex is a normal space with the spherical fibration given by the Spivak normal bundle construction.

A Poincaré complex of formal dimension $n$, that is presented as an $n$-dimensional simplicial complex, will be called a triangulated Poincaré complex.

---

*AMS (MOS) subject classifications* (1970). Primary 57B10, 55F25; Secondary 56C20.

*Key words and phrases.* Poincaré duality space, normal space, normally embedded subcomplex, Poincaré embedding.

*Notation.* If $G$ is an abelian group, $FG$ will denote the free abelian group obtained by dividing out by the torsion subgroup.

DEFINITION 2.1. Let $X$ be a triangulated Poincaré complex, $K$ a full subcomplex of $X$, and $\nu(k)$ a simplicial neighborhood of $K$ obtained by deriving $X$ near $K$; then $K$ is a *normally embedded subcomplex* of $X$ if the image of the fundamental class $[X]$ in $F(H_n(\nu(K)), \partial\nu(K)) \cong F(H_n(X, X - \nu(K)))$ induced by $c$: $X \longrightarrow X(X - \nu(K))$ is a generator.

If a group $G$ acts simplicially on $X$ with $K$ an invariant subcomplex, then an obvious modification of the above defines $K$ as an *equivariantly normally embedded subcomplex.* In the usual way one can also define a *normally embedded subpair* of a triangulated Poincaré pair.

EXAMPLE 2.2. Let $P$ be a Poincaré complex of formal dimension $\geqslant 5$; then there is a triangulated Poincaré complex $Q$ homotopy equivalent to $P$ such that $Q \times Q$ can be triangulated so that the diagonal is a *normally embedded subcomplex.* This follows since $Q$ can be chosen to be a simplicial $n$-cycle (for example by using a handle decomposition [2]).

We note the following lemma.

LEMMA 2.3. *Suppose $K$ is a normally embedded subcomplex of a triangulated Poincaré complex $X$, and $f$: $K \longrightarrow L$ a homotopy equivalence, $L$ a simplicial complex. Then $X$ can be modified by a homotopy equivalence so $K$ contains $L$ as a normally embedded subcomplex.*

PROOF. If $\nu(k)$ is the simplicial neighborhood of $K$, we can excise it and replace it by the mapping cylinder of $\partial\nu(k) \longrightarrow \nu(k) \longrightarrow K \longrightarrow L$.

COROLLARY. *If in Definition* 2.1, *$K$ is itself a Poincaré complex of formal dimension $\geqslant 5$, we can suppose $K$ is presented as a union of handles.*

PROOF. According to Jones [2] any Poincaré complex of dimension $\geqslant 5$ has a handle decomposition.

3. **The main theorem.** The main result of this paper is the following.

THEOREM 3.1. *Let $X$ be a triangulated Poincaré complex of dimension $n$, and suppose $K$ is a normally embedded subcomplex which is a triangulated Poincaré complex of dimension $k \geqslant 5$, $k < n$. Then the inclusion $K \longrightarrow N$ is homotopic to a Poincaré embedding.*

We shall outline the main idea of the proof (details will appear elsewhere

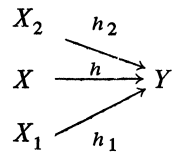
[6]). Let $\nu(K)$ be the simplicial neighborhood of $K$ in $X$. Suppose we can find a spherical fibration $\xi$: $E \longrightarrow K$ of rank $(n-k)$. $E$ is then a normal space of formal dimension $(n-1)$. Furthermore one can show that $\partial\nu(K)$ is also a normal space of formal dimension $(n-1)$. So if we can find a normal map $h$: $\partial\nu(k) \longrightarrow E$ we can replace $\nu(K)$ by $M_h \cup_E M_\xi$ in $X$. ($M_h$, $M_\xi$ are the mapping cylinders of $h$ and $\xi$ respectively.) It then follows from Wall [5, Proposition 2.7(ii)] that the pairs $((X-\nu(K)) \cup M_h, E)$, and $(M_\xi, E)$ give the required splitting of $X$. The construction of $E$ and the normal map $h$ are performed by a fairly straightforward induction using a handle decomposition of $K$.

With a simple modification of the proof, one obtains

COROLLARY. *Let $X$ be a Poincaré complex of formal dimension $n \geqslant 5$ and suppose $X \times X$ is triangulated as a $2n$-complex so that the diagonal is an equivariantly normally embedded subcomplex, where $Z_2$ acts on $X \times X$ by exchange of factors; then there is an equivariant splitting $b$: $X \times X \longrightarrow (M, \partial N) \cup (N, \partial N)$ with $N$ homotopy equivalent to the diagonal and $Z_2$ acting freely on $M$. The inclusion $\partial N \longrightarrow N$ is homotopic to a spherical fibration, the tangent bundle associated to the given triangulation.*

The bundle obtained appears to depend on the triangulation; this will be discussed elsewhere [6].

4. **The Bredon theorem.** In this section we indicate briefly how our results may be applied to the Bredon theorem mentioned in the introduction. In a manner parallel to Jones [2], we define a category $\mathrm{CW}_p$, whose objects are finite CW-complexes and whose morphisms are all continuous maps. A homotopy in $\mathrm{CW}_p$ from $h_1$: $X_1 \longrightarrow Y$ to $h_2$: $X_2 \longrightarrow Y$ is a commutative diagram[1]




where each vertical map is a $Z_p$-homotopy equivalence; that is, an isomorphism of fundamental groups and $Z_p$-homotopy. A finite CW-complex $X$ is a

[1] ADDED IN PROOF. In [6] it will be shown that the bundle is independent of the triangulations.

$CW_p$-Poincaré complex if it satisfies Poincaré duality for $Z_p$ coefficients and a $CW_p$-Poincaré embedding is defined in the natural way. A $CW_p$-normal space is a CW-complex $X$ together with a $Z_p$-spherical fibration $\xi$ and map $p\colon S^{n+k} \longrightarrow T(\xi)$. As before a simplicial complex of dimension $n$ which is a $CW_p$-Poincaré complex of formal dimension $n$ is called a triangulated $CW_p$-Poincaré complex. Again Jones' work implies that a $CW_p$-Poincaré complex of dimension $\geqslant 5$ has a handle decomposition.

DEFINITION 4.1. Let $X$ be a triangulated $CW_p$-Poincaré complex and $K$ a subcomplex. Then $K$ is a normally embedded subcomplex if $K$ has a simplicial neighborhood $\nu(K)$ in some derived neighborhood of $X$ near $K$ such that the image of the fundamental class $[X] \in H_n(X, Z_p)$ under the map $H_n(X, Z_p) \longrightarrow H_n(\nu(K), \partial\nu(K); Z_p)$ is nonzero. If $X$ has a group action and $K$ is invariant, then we can define *equivariantly normally embedded* in the obvious way.

One can then modify the proof of Theorem 3.1 to obtain

THEOREM 4.2. *Let $X$ be a triangulated $CW_p$-Poincaré complex of dimension $n$, and suppose $Z_p$ acts simplicially on $X$ with fixed point set $F$. Then for each component $F_0$ of $F$ that is a $CW_p$-Poincaré complex of formal dimension $\geqslant 5$ and is equivariantly normally embedded; this inclusion $F_0 \longrightarrow X$ is homotopic to an equivariant $CW_p$-Poincaré embedding.*

To show the applicability of this to the Bredon theorem we need the following results.

PROPOSITION 4.3. *Let $X$ be a $CW_p$-Poincaré complex of formal dimension $n$, and suppose $Z_p$ acts simplicially on $X$ with fixed point sets $F = F_1 \cup F_2 \cup \cdots \cup F_r$, each $F_i$ a $CW_p$-Poincaré complex of formal dimension $f_i \geqslant 3$. Then the $(r+1) - \operatorname{ad}(X; F_1, \cdots, F_r)$ is homotopy equivalent to an $(r+1) - \operatorname{ad}(X'; F_1', \cdots, F_r')$ of triangulated $CW_p$-Poincaré complexes.*

PROPOSITION 4.4. *Let $X$ be a triangulated $CW_p$-Poincaré complex of dimension $n$, and suppose $Z_p$ acts simplicially on $X$ with fixed point set $F$. Then each component $F_0$ of the fixed point set is equivariantly normally embedded.*

The first of these propositions is proved by constructing an appropriate model for the quotient space $X/Z_p$, and the second uses a spectral sequence argument following the method of proof of Bredon's theorem. We thus obtain

THEOREM 4.5. *Let $X$ be a $CW_p$-Poincaré complex represented by a simplicial complex on which $Z_p$ acts simplicially. Then if the formal dimension of each component of the fixed point set (as a $CW_p$-Poincaré complex) is $\geqslant 5$, $X$ is equivariantly homotopy equivalent to a triangulated $CW_p$-Poincaré complex $Y$, on which $Z_p$ acts and such that the fixed point set is $CW_p$-Poincaré embedded.*

## REFERENCES

1. G. Bredon, *Fixed point sets of actions on Poincaré duality spaces*, Topology **12** (1973), 159–176.

2. L. Jones, *Patch spaces: A geometric representation for Poincaré spaces*, Ann. of Math. (2) **97** (1973), 306–343. MR **47** #4269.

3. N. Levitt, *On the structure of Poincaré duality spaces*, Topology **7** (1968), 369–388. MR **40** #2089.

4. F. Quinn, *Surgery on Poincaré and normal spaces*, Bull. Amer. Math. Soc. **78** (1972), 262–267. MR **45** #6014.

5. C. T. C. Wall, *Surgery on compact manifolds*, Academic Press, New York, 1970.

6. J. P. E. Hodgson, *Embedded subcomplexes of Poincaré complexes* (in preparation).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19104

*Current address*: Department of Mathematics, Adelphi University, Garden City, Long Island, New York 11530