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OBSTRUCTIONS TO TRANSVERSALITY FOR COMPACT LIE GROUPS

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Throughout G is a compact Lie group which is topologically cyclic with dense generator g. Let N and M be smooth G manifolds without boundary and $Y \subset M$ a closed invariant submanifold. All manifolds are oriented and G preserves orientation. Let $f: N \to M$ be a proper G map. When is f properly G homotopic to a map γ which is transverse regular to $Y \subset M$, written $\gamma \pitchfork Y$? We introduce obstructions which show that transversality is a global phenomena in contrast to the case $G=1$ where everything is local and trivial.

Without loss of generality, we may assume that $f^g: N^g \to M^g$ is transverse to Y^g and set $X^g = (f^g)^{-1}(Y^g)$. For each oriented real G vector bundle v over X^g such that the G representation on each fiber of v has no trivial factor and g preserves orientation on each fiber, let $\lambda_\pm(v)$ be the $\pm$ eigenbundles of the canonical involution τ on $\lambda(v \otimes C) = \sum \lambda^i(v \otimes C)$ constructed from the orientation and an inner product on v. Let $\lambda_{-1}(v \otimes C) = \sum (-1)^i \lambda^i(v \otimes C)$, $I^{X^g} \in K_G(TX^g)$ be the index class of X^g, i.e. the symbol of the operator D^+. See [**1**, p. 575]. Let $\mathscr{P} \subset R(G)$ be the prime ideal of characters $\{X \in R(G) \mid X(g)=0\}$ and

$$\text{(i)} \qquad \mathscr{B}(v) = \frac{\lambda_+(v) - \lambda_-(v)}{\lambda_{-1}(v \otimes C)} \cdot I^{X^g} \in K_G(TX^g)_{\mathscr{P}}.$$

Let $f: X \to Y$ be a G map. If f is an embedding there is a homomorphism $f!: K_G(TX) \to K_G(TY)$ [**1**]. By taking the product of Y with a real G module and using the Thom isomorphism for complex G vector bundles, we may assume that $f!$ is defined for any map f and denote it by f_*. The normal bundle of Y in M is denoted by $\nu(Y, M)$. Its restriction to Y^g has a splitting

$$\text{(ii)} \qquad (i^g)^* \nu(Y, M) = \nu(Y, M)^g + \nu_2(Y, M),$$

where $\nu(Y, M)^g$ is the subbundle of points fixed by g and $i^g: Y^g \to Y$ is

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the inclusion. Let $j^g: X^g \to N^g$ be the inclusion and define $\nu = \nu(f)$ by

(iii) $$\nu + (f^g)^* \nu_2(Y, M) = (j^g)^* \nu(N^g, N)$$

and set

$$B_G = \mathscr{B}(\nu(f)) \in K_G(TX^g)_{\mathscr{P}}.$$

The inclusions of TN^g in TN and TY^g in TY are denoted by Th^g and Ti^g. Let $(\mathrm{Id}_G^{X^g})_{\mathscr{P}}: K_G(TX^g)_{\mathscr{P}} \to R(G)_{\mathscr{P}}$ be the localization of the Atiyah-Singer index homomorphism. The group of connected components of G is denoted by $\Pi_0(G)$. Define

(iv) $$g(f) = (\mathrm{Id}_G^{X^g})_{\mathscr{P}}(B_G) \in R(G)_{\mathscr{P}}/R(\Pi_0(G)),$$

(v) $$l(f) = \lambda_{-1}(\nu(N^g, N)^{\cdot} \otimes C) \cdot j^g_*(B_G) \in K_G(TN^g)_{\mathscr{P}}/(Th^g)^* K_G(TN),$$

(vi) $$\mathscr{O}(f) = \lambda_{-1}(\nu(Y^g, Y) \otimes C) f^g_*(B_G) \in K_G(TY^g)_{\mathscr{P}}/(Ti^g)^* K_G(TY).$$

THEOREM 1. *If $f: N \to M$ is properly G homotopic to γ and $\gamma \pitchfork Y$, then $g(f)$, $l(f)$ and $\mathscr{O}(f)$ are zero.*

PROOF. Suppose $f \pitchfork Y$ and $X = f^{-1}(Y)$. Then $g(f) = \mathrm{Sign}(G, X) \in R(\Pi_0(G))$; moreover, $\mathscr{O}(f) = (Ti^g)^* f_*(I^X)$, where $I^X \in K_G(TX)$ is the index class of X. Similarly one sees that $l(f) = 0$.

The notion of fiber homotopy equivalence is extended to the category of G vector bundles as follows: Let N and M be two (real) G bundles over a G space Y. A G map $\omega: N \to M$ is called a *quasi-equivalence* if ω is proper, fiber preserving and degree 1 on fibers. The notion of normal map is extended to the category of smooth, closed G manifolds as follows: A *normal G map $f: X \to Y$* consists of a triple $[X, f, F]$ where $f: X \to Y$ is a G map of degree 1 and F is a bundle map $F: TX + f^*(N) \to TY + M$ covering f for some pair of G bundles N and M over Y. When $G = 1$ the set of normal cobordism classes of normal maps to Y and the set of fiber homotopically equivalent bundles with appropriate equivalence relation are in 1-1 correspondence. Transversality provides the correspondence. For general G and quasi-equivalence $\omega: N \to M$, the obstructions $g(\omega)$, $l(\omega)$ and $\mathscr{O}(\omega)$ to making $\omega \pitchfork Y$ give obstructions to converting a quasi-equivalence to a normal G map.

EXAMPLE 1. Let $G = S^1$ with dense generator t. Let N and M be two complex S^1 vector bundles over a closed S^1 manifold Y. To simplify the formula, we assume $N^t = M^t = Y^t$. The restrictions $\tilde{N}$ and $\tilde{M}$ of N and M to Y^t have real splittings $\tilde{N} = \sum_{n>0} N_n$, $\tilde{M} = \sum_{n>0} M_n$ where, e.g., N_n is the subbundle on which t acts by multiplication by t^n. Similarly, $\nu(Y^t, Y)$ has such a splitting. Let

(vii) $$A(t, \tilde{N}) = \prod_{n>0} \prod_j \frac{t^n e^{x_j} + 1}{t^n e^{x_j} - 1}(N_n) \in H^*(Y^t, C),$$

where the elementary symmetric functions of the $x_j = x_j(N_n)$ are the Chern classes of N_n,

(viii) $$L'(TY^t) = \prod \frac{x_i}{\tanh(x_i/2)}(TY^t) \in H^*(Y^t, C),$$

where the elementary symmetric functions of the x_i^2 are the Pontrjagin classes of Y^t.

The ring $R(S^1)_{\mathscr{P}}$ is contained in the field $Q(t)$ of rational functions of t. The obstruction $g(\omega) \in Q(t)/Z$ $(Z = R(1))$ is given by the rational function

(ix) $$g(\omega)(t) = \left\langle A(t, \nu(Y^t, Y)) \frac{A(t, \tilde{N})}{A(t, \tilde{M})} L'(TY^t), [Y^t] \right\rangle$$

where $\langle \alpha, [Y^t] \rangle$ denotes evaluation of the cohomology class α on the orientation class $[Y^t] \in H_*(Y^t, C)$. Observe that the obstruction $g(\omega)$ does not depend on ω. Essentially the reason is that S^1 is connected. (Compare (ix) with [**1**, (7.7)].)

EXAMPLE 1′. As a very special illustration of (ix), take Y to be a point and N and M the complex two-dimensional S^1 modules $N = t^p + t^q$ and $M = t + t^{pq}$, $(p, q) = 1$, where $t \in S^1$ acts with eigenvalues t^p and t^q, respectively, t and t^{pq}. We view N and M as S^1 vector bundles over Y. Choose positive integers a and b such that $-ap + bq = 1$. Let $z = (z_0, z_1)$ be complex coordinates of a point $z \in N$ and set $\omega_0(z) = (\bar{z}_0^a z_1^b, z_0^q + z_1^p)$. Then ω_0 is a proper S^1 map and has degree 1; moreover,

(x) $$g(\omega_0)(t) = \frac{(t^p + 1)(t^q + 1)(t - 1)(t^{pq} - 1)}{(t^p - 1)(t^q - 1)(t + 1)(t^{pq} + 1)} \in \frac{Q(t)}{Z}.$$

PRODUCT LEMMA 3. *Let N and M be two complex G modules viewed as G bundles over a point, and $\omega : N \to M$ a quasi-equivalence. Then ω induces a quasi-equivalence $\tilde{\omega} : Y \times N \to Y \times M$ for any closed G manifold Y and*

$$g(\tilde{\omega}) = \operatorname{Sign}(G, Y) \cdot g(\omega).$$

COROLLARY 4. *Let Y be a closed S^1 manifold with* $\operatorname{Sign}(Y) = \operatorname{Sign}(1, Y) \neq 0$. *Let $\omega_0 : N \to M$ and $\tilde{\omega}_0 : Y \times N \to Y \times M$ be as above. Then $g(\tilde{\omega}_0) \neq 0$.*

EXAMPLE 2. Let $\omega : N \to M$ be a quasi-equivalence of G bundles over Y. Assume $N^g = M^g = Y^g$ consists of q isolated points. Then $K_G(TY^g)_{\mathscr{P}} = \prod_{j=1}^q R(G)_{\mathscr{P}}$ and the jth component of $\mathscr{O}(\omega)$ is

$$\mathscr{O}(\omega)_j = \frac{\lambda_{-1}(M_j \otimes C)}{\lambda_{-1}(N_j \otimes C)} \cdot \frac{\lambda_+(TY_j) - \lambda_-(TY_j)}{\lambda_+(M_j) - \lambda_-(M_j)},$$

where M_j, N_j and TY_j denote the representations of G defined by restricting N, M and TY to the jth isolated fixed point.

In order to illustrate ideas for closed manifolds, observe that any quasi-equivalence $\omega: N \to M$ induces a G map $\omega^+: N^+ \to M^+$ of the one point compactifications. In particular, take the N and M of Example 1′ and $\omega = \omega_0$. Then N^+ and M^+ are smooth 4 spheres. Take $Y = (M^+)^{S^1}$. One finds that $\mathcal{O}(\omega_0^+) \neq 0$.

Contributions of subgroups $H \subset G$. Each subgroup of G can be used to generate new obstructions via the following observation: If $f: N \to M$ is transverse to $Y \subset M$ then $f^H: N^H \to M^H$ is transverse to Y^H for each $H \subset G$; moreover, f^H is a G/H map. This means that if $\alpha_{\bar{G}}$ is any transversality obstruction defined for all topologically cyclic groups $\bar{G}$, then $\alpha_{G/H}(f^H)$ is a transversality obstruction for the G map f, i.e., $\alpha_{G/H}(f^H) = \mathscr{B}_G(f)$ is an obstruction for f. Actually each component of M^H contributes an obstruction.

EXAMPLE 3. $G = S^1$. Let $\Omega = t^p + t^p + t^p + t^0$ be the complex 4-dimensional S^1 module where $t \in S^1$ acts with eigenvalues t^p, t^p, t^p and t^0. Let $Y = P(\Omega)$ be the space of complex lines in Ω. Then Y is an S^1 manifold in an obvious way and if $\omega_0: N \to M$ is the quasi-equivalence of Example 1′, then $\tilde{\omega}_0: Y \times N \to Y \times M$ and $g_{S^1}(\omega_0) = 0$ by the Product Lemma. On the other hand, $(Y \times M)^{Z_p}$ consists of two components and each contributes an obstruction

$$g_{S^1/Z_p}(\tilde{\omega}_0{}^{Z_p})_i = \frac{t^1 + 1}{t^1 - 1}\frac{t^q - 1}{t^q + 1} \in \frac{Q(t)}{Z}$$

for $i = 1, 2$. Here S^1/Z_p is identified with S^1 with representation ring $Z[t, t^{-1}]$.

Equation (iii) provides the basis for an obstruction theory for G transversality, but this and other details of G transversality including the case of finite isotropy groups will appear elsewhere.

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