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A FIXED POINT THEOREM FOR MULTIVALUED NONEXPANSIVE MAPPINGS IN A UNIFORMLY CONVEX BANACH SPACE

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Let C be a nonempty weakly compact convex subset of a Banach space X, and $\mathscr{C}(C)$ be the family of nonempty compact subsets of C equipped with the Hausdorff metric. Let $T: C \to \mathscr{C}(C)$ be a nonexpansive mapping, i.e. for each $x, y \in C$,

$$H(T(x), T(y)) \leqq \|x - y\|,$$

where $H(A, B)$ denotes the Hausdorff distance between A and B. A point $x \in C$ is called a fixed point of T if $x \in Tx$. Fixed point theorems for such mappings T have been established by Markin [**11**] for Hilbert spaces, by Browder [**2**] for spaces having weakly continuous duality mapping, and by Lami Dozo [**7**] for spaces satisfying Opial's condition. Lami Dozo's result is also generalized by Assad and Kirk [**1**]. By making use of Edelstein's asymptotic center [**4**], [**5**], we are able to prove Theorem 1. Let C be a closed convex subset of a uniformly convex Banach space and let $\{u_i\}$ be a bounded sequence in C. The asymptotic center x of $\{u_i\}$ in (or with respect to) C is the unique point in C such that

$$\limsup_i \|x - u_i\| = \inf\left\{\limsup_i \|y - u_i\| : y \in C\right\}.$$

The number $r = \inf\{\limsup_i \|y - u_i\| : y \in C\}$ is called the asymptotic radius of $\{u_i\}$ in C. Existence of the unique asymptotic center is proved by Edelstein in [**5**]. Results on ordinal numbers used here may be found in [**13**].

THEOREM 1. *Let X be a uniformly convex Banach space and C be a closed convex bounded nonempty subset of X. Let $T: C \to \mathscr{C}(C)$ be a nonexpansive mapping from C into the family of nonempty compact subsets of*

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C (equipped with the Hausdorff metric). Then T has a fixed point, i.e. there exists $x \in C$ with $x \in Tx$.

PROOF. Let a be a point in C fixed throughout the proof. Let $\{\lambda_m\}$ be a decreasing sequence of positive numbers and $\lim \lambda_m = 0$. For each m, the mapping $T_m: C \to \mathscr{C}(C)$ defined by $T_m(x) = \lambda_m a + (1-\lambda_m)Tx$ is a contraction mapping and hence has a fixed point x_m (Nadler [**12**]). Thus $x_m \in \lambda_m a + (1-\lambda_m)Tx_m$, and there exists $y_m \in Tx_m$ with $x_m = \lambda_m a + (1-\lambda_m)y_m$. Since C is bounded, we have

$$\|x_m - y_m\| = \lambda_m \|a - y_m\| \to 0 \quad \text{as } m \to \infty.$$

To facilitate the later description, we define $i: \{x_m\} \to \{y_m\}$ by $i(x_m) = y_m$ for all m. We say that a sequence $\{x_n\}$ is an essential subsequence of $\{y_m\}$ if for some $N > 0$, $\{x_n\}_{n \geqq N}$ is a subsequence of $\{y_m\}$.

Define the sequence $\{x_m^{(0)}\}$ to be $\{x_m\}$, i.e. $x_m^{(0)} = x_m$ for each $m \geqq 1$. Let Ω be the first uncountable ordinal and β be a countable ordinal, i.e. $\beta < \Omega$. Suppose that $\{x_m^{(\alpha)}\}$ has been defined for every ordinal α less than β in such a way that $\{x_m^{(\gamma)}\}$ is an essential subsequence of $\{x_m^{(\delta)}\}$ whenever $\delta < \gamma < \beta$. We define $\{x_m^{(\beta)}\}$ as follows:

Case 1. β has an immediate predecessor, i.e., $\beta = \alpha + 1$ for some $\alpha < \Omega$. Let z_α be the asymptotic center of $\{x_m^{(\alpha)}\}$ in C. For each m, let $p_m \in Tz_\alpha$ be chosen such that

$$\|p_m - y_m^{(\alpha)}\| \leqq \|z_\alpha - x_m^{(\alpha)}\|,$$

where $y_m^{(\alpha)} = i(x_m^{(\alpha)})$; existence of such a p_m is a consequence of the nonexpansiveness of T and the compactness of Tz_α. Since Tz_α is compact and $\{p_m\} \subseteq Tz_\alpha$, there exists a convergent subsequence $\{p_{m_i}\}$ of $\{p_m\}$. We then define $\{x_m^{(\beta)}\}$ to be the sequence $\{x_{m_i}^{(\alpha)}\}$.

Case 2. β is a limit ordinal. Then there exists a strictly increasing sequence $\{\alpha_n\}$ of ordinal numbers such that $\alpha_n < \beta$ for each n and $\alpha_n \to \beta$, i.e. for every $\alpha < \beta$, there exists n such that $\alpha < \alpha_n < \beta$. By dropping a finite number of terms if necessary, we may assume that $\{x_m^{(\alpha_n)}\}$ is a subsequence of $\{x_m^{(\alpha_p)}\}$ whenever $p < n$. We then define $\{x_m^{(\beta)}\}$ to be the sequence constructed from $\{x_m^{(\alpha_n)}\}$ by the diagonal process, i.e., $\{x_m^{(\beta)}\} = \{x_m^{(\alpha_m)}\}$. Then $\{x_m^{(\beta)}\}$ is an essential subsequence of $\{x_m^{(\alpha_n)}\}$ for each n. Since $\alpha_n \to \beta$, $\{x_m^{(\beta)}\}$ is an essential subsequence of $\{x_m^{(\alpha)}\}$ whenever $\alpha < \beta$.

Hence $\{x_m^{(\alpha)}\}$ are defined for all $\alpha < \Omega$. Now for each $\alpha < \Omega$, we let r_α be the asymptotic radius of $\{x_m^{(\alpha)}\}$ in C. Since $\{x_m^{(\gamma)}\}$ is an essential subsequence of $\{x_m^{(\delta)}\}$ whenever $\delta < \gamma$, and since $r_\alpha \geqq 0$ for every $\alpha < \Omega$, the transfinite sequence $\{r_\alpha : \alpha < \Omega\}$ on the real line is decreasing and has lower bound 0. Let $s = \inf\{r_\alpha : \alpha < \Omega\}$. Then clearly $\lim\{r_\alpha : \alpha < \Omega\}$ exists and equals s. This can happen only if for some $\beta_0 < \Omega$, $r_\alpha = s$ for all α with $\beta_0 < \alpha < \Omega$. Let

α be a fixed ordinal with $\beta_0<\alpha<\Omega$. We shall show that the asymptotic center z_α of $\{x_m^{(\alpha)}\}$ is a fixed point of T.

From the way that $\{x_m^{(\alpha+1)}\}$ is constructed from $\{x_m^{(\alpha)}\}$, there exists a convergent sequence $\{p_m\}\subseteq Tz_\alpha$ with $\lim p_m=p \in Tz_\alpha$ such that

$$\|p_m - y_m^{(\alpha+1)}\| \leqq \|z_\alpha - x_m^{(\alpha+1)}\| \tag{1}$$

for all m, where $y_m^{(\alpha+1)}=i(x_m^{(\alpha+1)})$. Since $\{x_m^{(\alpha+1)}\}$ is a subsequence of $\{x_m^{(\alpha)}\}$, and $x_m^{(\alpha+1)}-y_m^{(\alpha+1)}\to 0$, we have from (1):

$$\begin{aligned}\limsup_m \|p - x_m^{(\alpha+1)}\| &= \limsup_m \|p - y_m^{(\alpha+1)}\| \\ &\leqq \limsup_m \|z_\alpha - x_m^{(\alpha+1)}\| \\ &\leqq \limsup_m \|z_\alpha - x_m^{(\alpha)}\| = r_\alpha = r_{\alpha+1}.\end{aligned}$$

It follows from the uniqueness of the asymptotic center that $p=z_{\alpha+1}$ and $z_\alpha=z_{\alpha+1}$, where $z_{\alpha+1}$ is the asymptotic center of $\{x_m^{(\alpha+1)}\}$ in C. Hence $z_\alpha= p \in Tz_\alpha$, completing the proof.

REMARK. Theorem 1 remains true if X is required only to be reflexive and uniformly convex in every direction [**6**], [**3**], since in such spaces the asymptotic center of a bounded sequence in a closed convex set is unique [**10**].

We do not know whether Theorem 1 is true when C is required only to be weakly compact and to have normal structure. For the application of asymptotic center under this setting, see [**8**] and [**9**].

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