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UNBOUNDED OPERATORS WITH SPECTRAL CAPACITIES

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The concept of spectral capacity introduced by C. Apostol in [1] and its relationship to decomposable operators [3] established by a theorem of C. Foiaş [4] are used for an investigation in the unbounded case.

Let $\mathfrak{S}(X)$ denote the family of subspaces (closed linear manifolds) of a Banach space X, and let $\mathfrak{F}$ and $\mathfrak{K}$ represent the collection of closed and compact subsets of the complex plane π, respectively. The superscript c stands for the complement.

1. Definition [1]. A spectral capacity in X is an application $\mathfrak{E}:\mathfrak{F}\to\mathfrak{S}(X)$ which satisfies the following conditions:

(i) $\mathfrak{E}(\varnothing)=\{0\}$, $\mathfrak{E}(\pi)=X$;

(ii) $\bigcap_{n=1}^{\infty}\mathfrak{E}(F_n)=\mathfrak{E}(\bigcap_{n=1}^{\infty}F_n)$, $\{F_n\}\subset\mathfrak{F}$;

(iii) for every finite open cover $\{G_i\}_{1\leqq i\leqq m}$ of $F\in\mathfrak{F}$, $\mathfrak{E}(F)=\sum_{i=1}^{m}\mathfrak{E}(F\cap\bar{G}_i)$.

In order to confine the present investigation to densely defined operators on X, the following additional constraint on the spectral capacity is needed:

2. Definition. A spectral capacity $\mathfrak{E}$ will be referred to as regular if the linear manifold

$$X_0=\{x\in\mathfrak{E}(K):K\in\mathfrak{K}\}$$

is dense in X.

3. Definition. A linear operator $T:D(T)\ (\subseteq X)\to X$ is said to possess a regular spectral capacity $\mathfrak{E}$ (abbrev. $T\in\mathfrak{T}(\mathfrak{E})$) if it is closed, has a nonvoid resolvent set and satisfies the following conditions:

(iv) $\mathfrak{E}(K)\subseteq\mathfrak{D}(T)$ for all $K\in\mathfrak{K}$;

(v) $T(\mathfrak{E}(F)\cap\mathfrak{D}(T))\subseteq\mathfrak{E}(F)$ for all $F\in\mathfrak{F}$;

(vi) the restriction $T_F=T|\mathfrak{E}(F)\cap\mathfrak{D}(T)$ has the spectrum $\sigma(T_F)\subseteq F$, $F\in\mathfrak{F}$.

4. Theorem. *Given $T\in\mathfrak{T}(\mathfrak{E})$. For every $K\in\mathfrak{K}$, the restriction $T_K=T|\mathfrak{E}(K)$ is a (bounded) decomposable operator on $\mathfrak{E}(K)$ possessing the*

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spectral capacity $\mathfrak{E}_K$ *defined by*

(1) $$\mathfrak{E}_K(F) = \mathfrak{E}(K \cap F) \quad \textit{for all } F \in \mathfrak{F}.$$

In the proof it is shown that T_K is bounded by the closed graph theorem and $\mathfrak{E}_K$, as defined by (1), is a spectral capacity for T_K.

A property which is instrumental for the subsequent study of operators in $\mathfrak{F}(\mathfrak{E})$ is expressed by the following

5. THEOREM. *Let* $T \in \mathfrak{T}(\mathfrak{E})$ *and* $K \in \mathfrak{K}$. *The following statements are equivalent*:

(i) $x \in \mathfrak{E}(K)$;

(ii) *there exists an X-valued function* $\tilde{x}$ *analytic on* K^c *satisfying the equation*

$$(\lambda - T)\tilde{x}(\lambda) = x \quad \textit{for all } \lambda \in K^c.$$

The implication (i)$\Rightarrow$(ii) of the proof is based on the single-valued extension property of a decomposable operator. (ii)$\Rightarrow$(i) is proved first for an $x \in X_0$ with the help of a result by C. Foiaş [**4**]:

$$\{y \in \mathfrak{E}(L): \sigma_{T_L}(y) \subseteq K\} = \mathfrak{E}(K) \quad \text{where } L(\supset K) \in \mathfrak{K}.$$

Next, for $x \notin X_0$, the density of X_0 in X and the closeness of $\mathfrak{E}(K)$ complete the proof.

6. THEOREM. *Every* $T \in \mathfrak{T}(\mathfrak{E})$ *has a unique regular spectral capacity.*

In the first stage of the proof, the application of Theorem 5 shows that any two regular spectral capacities $\mathfrak{E}$ and $\mathfrak{E}_1$ of T agree on $\mathfrak{K}$. Next the property expressed by Definition 2 implies that $\mathfrak{E}(F)=\mathfrak{E}_1(F)$ for all $F \in \mathfrak{F}$.

7. THEOREM. *For every* $K \in \mathfrak{K}$, $\mathfrak{E}(K)$ *is a spectral maximal space of* $T \in \mathfrak{T}(\mathfrak{E})$.

The proof is performed with the help of Theorems 4 and 5.

8. THEOREM. *Given* $T \in \mathfrak{T}(\mathfrak{E})$. *For every* $x \in X$ *there exists a nonvoid open set* $U \subset \pi$ *and a sequence* $\{\tilde{x}_n\}$ *of X-valued functions analytic on* U, *with*

$$\lim_{n\to\infty} (\lambda - T)\tilde{x}_n(\lambda) = x \quad \textit{for all } \lambda \in U.$$

Again, the proof is obtained by an application of Theorem 5.

We redefine E. Bishop's concept of weak spectral manifold $\mathfrak{N}(F, T)$ [**2**, Definition 2] without the restriction of T being bounded as follows: Given $T: \mathfrak{D}(T)\ (\subseteq X) \rightarrow X$ and $F \in \mathfrak{F}$, $\mathfrak{N}(F, T)$ is the set of all $x \in X$ which have the property that for each $\varepsilon > 0$ there exists an X-valued function $\tilde{x}$ analytic on F^c such that $\|x-(\lambda-T)\tilde{x}(\lambda)\| < \varepsilon$, for all $\lambda \in F^c$.

A straightforward consequence of Theorem 8 is the following

9. COROLLARY. *Given* $T \in \mathfrak{T}(\mathfrak{E})$. *For every* $F \in \mathfrak{F}$,

$$\mathfrak{E}(F) = \mathfrak{N}(F, T).$$

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