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BOOK REVIEW

Infinite linear groups, B. A. F. Wehrfritz, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 76, Springer-Verlag, 1973, 231+xiii pp., $21.90.

In the present context "linear group" refers to any group with a faithful linear representation of finite degree n over a commutative field F. Equivalently, a linear group is a group isomorphic to a subgroup of the general linear group $\mathrm{GL}(n, F)$. Occasionally the term will be extended to include "linear groups" over a commutative ring.

Linear groups arise in many situations. The classical linear groups over finite fields are vital in the classification of finite simple groups. Linear groups occur as groups of rotations of lattices in three dimensions in a study of crystallographic groups. In the study of groups in general, linear groups occur very naturally as groups of automorphisms of certain classes of abelian groups or of groups with abelian factors. Free groups of countable rank and polycyclic groups have faithful linear representations over the integers, and simple proofs of the residual properties of these groups can be based on this fact. Often linear groups serve as readily constructed examples and counterexamples in group theory.

The subject matter and techniques of the theory of group representations and the study of linear groups have much in common, but the emphases are different. In the former case one is trying to say something about the class of all representations of a given group, whilst in the second case one is usually studying a single representation or a small subset of representations of a group. Since very little seems to be known about the class of all representations of an infinite (discrete) group, the second point of view is of particular interest in the theory of infinite groups, and in this connection the linearity of a group is used as a kind of finiteness condition. Some classical examples are the following: A linear group of finite exponent is finite (Burnside, 1905), and a linear locally solvable group is solvable (Zassenhaus, 1938).[1] Furthermore, restrictions on the degree (and sometimes on the characteristic of the underlying field) lead to results such as: A periodic linear group G of degree n over a field of characteristic 0 has an abelian normal subgroup A whose index $|G:A|$ is bounded by a function depending only on n (Jordan, 1878 and Schur, 1905).

In the book under review, Wehrfritz makes a systematic survey of

[1] In some cases dates are only approximate and are quoted to give an idea of the development of the subject. For more detailed information the reader should refer to the book itself.

what is now known about the answer to the question: If an infinite group has a faithful linear representation, what does this tell us about the structure of the group? It is a worthwhile piece of work because in the last fifteen to twenty five years a large number of results have been discovered, and much of this material was only available in scattered papers; often isolated results proved by people who needed them for some particular application. As far as the reviewer knows, the only books dealing to any extent with these results in linear groups are: a book by D. A. Suprunenko, *Solvable and nilpotent linear groups*, Transl. Math. Monographs, vol. 9, Amer. Math. Soc., 1963 (original Russian edition, 1958), which is perhaps to be superseded by his book, *Matrix groups* to be published in Russian in 1974; the book by B. I. Plotkin, *Groups of automorphisms of algebraic systems*, Wolters-Noordhoff, 1972 (original Russian edition, 1966), whose main concern is a much more general class of groups but which has a large section on linear groups; and the reviewer's book, *Structure of linear groups*, van Nostrand-Reinhold, 1972, which covers a variety of topics on both finite and infinite linear groups. However, none of these latter books attempts to give the thorough survey of infinite linear groups made in Wehrfritz' book. Wehrfritz' book itself is based on his Queen Mary College notes published in 1969, but the earlier material has been revised and more than half the material of the present book is completely new.

What then is known about infinite linear groups? The development of the book is as follows. A preliminary chapter deals with subgroups of $GL(n, F)$ and the classical theorems of Maschke, Schur, Burnside and Clifford. The second chapter deals with the linearity of certain classes of groups: Mal'cev's theorem (1940) characterizing the abelian groups which are linear of degree n; the theorem due to L. Auslander and Swan (1967) that each polycyclic group is a linear group over the ring of integers; the linearity of free groups; Magnus' theorems (1939) on the linearity of free metabelian and free nilpotent groups; and theorems of Nisnevič (1940), Vapne (1970) and Wehrfritz on the linearity of free products of wreath products. The general problem of characterizing linearity by group theoretic properties is still unsolved, but the chapter includes the important theorem of Mal'cev (1940) (essentially a theorem in model theory) which states: A group is a linear group of degree n over a field of characteristic $p \geqq 0$ if and only if each finitely-generated subgroup is linear of the same degree n over a field of the same characteristic p. The third chapter deals with solvable linear groups. This class of groups is fairly well understood. At a fairly early stage, work of Zassenhaus (1938) and Mal'cev (1951) showed that a locally solvable linear group of degree n is in fact solvable with its derived length bounded by a function depending only on n, and is of the form nilpotent-by-abelian-by-finite. Suprunenko later described (1957) the structure of primitive solvable subgroups of GL(n, F) in some detail, and this has been used by the reviewer and others to obtain more detailed information about solvable linear groups.

Chapter 4 is a key chapter. It deals with finitely-generated linear groups, or more generally with subgroups of GL(n, R), where R is a finitely-generated integral domain. In this chapter we find the ideas of Mal'cev (1940) as developed and extended by Platonov (1968), Gruenberg (1966) and others. The basic technique depends on the fact that a ring R of this type is Noetherian and has the property that R/M is a finite field for each maximal ideal M. Therefore, one can "approximate" GL(n, R) by the finite linear groups GL(n, R/M), and this forms a powerful tool for studying the subgroups of GL(n, R). For example: Each finitely generated linear group over a field of characteristic 0 has a torsionfree subgroup of finite index (first proved by A. Selberg in a different manner in 1960); and large classes of theorems for finite groups, such as solvability and nilpotence criteria, can be transferred directly to valid theorems for finitely generated linear groups.

Concepts and results from the theory of algebraic geometry have come to play an important role in the study of linear groups. Some of the simpler results form the material of Chapters 5 to 7. The fifth chapter deals with elementary properties of the Zariski topology with respect to the subgroups of GL(n, F), and Chapter 6 shows that the quotient G/N of a subgroup G of GL(n, F) is a linear group over F whenever N is a normal subgroup of G which is closed in G under the Zariski topology. Since factor groups of linear groups are not usually linear, this latter result is frequently useful. Chapter 7 considers the Jordan decomposition of a nonsingular matrix x into a commuting product of a diagonalizable matrix x_d and a unipotent matrix x_u. It includes a new proof (due to the author) that over an algebraically closed field, x_d and x_u lie in the Zariski closure of the subgroup generated by x; from this it follows that a Zariski closed locally nilpotent linear group is a direct product of its diagonalizable and unipotent subgroups. This chapter also includes a characterization of complete reducibility of solvable linear groups essentially due to the reviewer (1964).

Chapters 8 and 11 deal with theorems of Garaščuk (1960), Gruenberg (1968) and the author (1971) concerning the central height (that is, the length of a transfinitely continued upper central series) of a linear group, and the hypercyclic analogue called the "paraheight". The main results are that both the central height and the paraheight of a linear group of degree n over a field of characteristic $p \geqq 0$ have bounds of the type $\omega + d$, where ω is the first infinite ordinal and d is a finite constant depending only on n and p.

Periodic linear groups are dealt with in Chapters 9 and 12. These are now a well-studied class of groups with the basic theorems going back to Jordan and Schur. In particular, such groups are locally finite, and over fields of characteristic 0 are abelian-by-finite. Because of this latter result the major interest now lies in the case where the underlying field has characteristic $p>0$. In this case D. J. Winter (1968) showed that such a group is at least nilpotent-by-countable, and Brauer and Feit (1966) have shown that if the Sylow p-group is finite then the group is abelian-by-finite. Generally a large class of results about finite groups continue to hold for periodic linear groups. For example, Sylow theorems, transfer theorems, and theorems of Thompson and Glauberman all have natural analogues.

Perhaps some of the deepest results of the theory are contained in Chapter 10. In this chapter the author considers some theorems of Platonov and Tits (barely sketching the proofs) and develops some of the consequences. With a proof based on the classification of semisimple algebraic groups, Platonov showed (1967) a linear group which satisfies a nontrivial law is solvable-by-finite. This result enabled Merzljakov to answer (1967) affirmatively for linear groups questions of P. Hall on the finiteness of certain verbal and marginal subgroups. In 1972, Tits proved that every linear group is either solvable-by-locally-finite or else contains a free subgroup of rank 2. One consequence of Tits' theorem is that a Noetherian linear group must be solvable-by-finite; it is still unknown whether all Noetherian groups are solvable-by-finite. Many of the other results of Chapter 10 (for example, the theorem of Platonov quoted above) can be deduced from Tits' theorem.

The book concludes with a chapter showing how many of the earlier results can be extended in a routine manner to "linear groups" over commutative Noetherian rings, and an appendix giving a useful summary of concepts and theorems from the theory of algebraic groups which seem important in the study of linear groups.

The Bibliography (supplemented by Suggestions for Further Reading) lists only material referred to in the text. For a more extensive list one could refer to the *Index of papers in group theory* recently published by the American Mathematical Society and the bibliographies of surveys issued periodically by the Russian mathematicians. The most recent of the latter surveys is Yu. I. Merzljakov, *Linear groups*, Itogi Nauki-Ser. Mat. (1970) (=J. Soviet Math. 1 (1973), 571–593).

Wehrfritz' book is carefully and attractively written. As well as giving a thorough, well-documented survey of its field, it will appeal to anyone with a good background in group theory seeking a clear and readable introduction. *Infinite linear groups* is written in the best tradition of the Ergebnisse series, and it should be the standard reference in its field for some time to come.

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