BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 5, September 1974

GLOBAL BIFURCATION THEOREMS FOR NONCOMPACT OPERATORS

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Communicated by Michael Golomb, February 24, 1974

1. **Introduction.** The first general existence theorem for bifurcation points was obtained by Krasnoselski [**1**]. He considered the equation $u=\lambda Lu+H(\lambda, u)$ in a real Banach space $\mathscr{B}$ where L and H are compact, and H is $o(\|u\|)$ uniformly on each bounded λ interval for small u. In this situation he proved that if λ is a characteristic value of L having odd multiplicity, then $(\lambda, 0)$ is a bifurcation point in $R\times\mathscr{B}$. Much more recently, Rabinowitz [**2**] considered the same problem and, using a Leray-Schauder degree argument, obtained a two-fold alternative for the global behavior of these bifurcation branches.

This paper extends the results of Krasnoselski and Rabinowitz to a much larger class of operator equations. First to be considered is the equation

$$Lu = \lambda u + H(\lambda, u) \tag{1}$$

in a real Hilbert space $\mathscr{H}$, where H is as above and L is selfadjoint (bounded or unbounded). In this case, each isolated eigenvalue of L having odd multiplicity is a bifurcation point possessing a continuous branch. Moreover, an alternative theorem on the global behavior of these branches is obtained.

By use of similar arguments these results for selfadjoint operators are extended to a general class of linear operators in a real Banach space $\mathscr{B}$.

2. **The selfadjoint operators.** In this section all work is in a real Hilbert space $\mathscr{H}$, L is a selfadjoint operator taking $\mathscr{H}$ into $\mathscr{H}$, and $H(\lambda, u)$ is a compact operator taking $R\times\mathscr{H}$ into $\mathscr{H}$ that is $o(\|u\|)$ uniformly on each bounded λ interval for small u.

Let $\mathscr{E}$ denote $R\times\mathscr{H}$ with the product topology. For $\mathscr{V}\subset\mathscr{E}$, a subcontinuum of $\mathscr{V}$ is a subset of $\mathscr{V}$ which is closed and connected in $\mathscr{E}$. The trivial solutions of (1) are the points $(\lambda, 0)$, and all other solutions are called nontrivial. Let $\mathscr{S}$ denote all nontrivial solutions of (1), and let $\mathscr{C}_{\lambda_0}$ denote the maximal subcontinuum of $\mathscr{S}\cup(\lambda_0, 0)$ containing $(\lambda_0, 0)$.

AMS (MOS) subject classifications (1970). Primary 47H15, 46N05.
Key words and phrases. Nonlinear operator equations, bifurcation.

For a subset A of R, $\mathscr{H}$, or $\mathscr{E}$, $\mathrm{Cl}(A)$ denotes its closure in the respective space. For $A\subset\mathscr{E}$, A_R denotes $\{\lambda|(\lambda,u)\in A$ for some $u\}$, and $A_{\mathscr{H}}$ denotes $\{u|(\lambda,u)\in A$ for some $\lambda\}$. By an isolated eigenvalue λ of L, we mean that λ is an eigenvalue of L and $\operatorname{dist}(\lambda, \operatorname{sp} L\backslash\lambda)>0$.

The following lemma is stated without proof.

LEMMA 1. *Suppose λ_0 is an isolated eigenvalue of L having finite multiplicity. Assume $\mathscr{C}_{\lambda_0}$ is bounded, $\mathrm{Cl}((\mathscr{C}_{\lambda_0})_R)\cap\operatorname{ess\,sp} L=\varnothing$, and $\mathscr{C}_{\lambda_0}\cap\{R\times\{0\}\}=(\lambda_0,0)$. Then $\mathscr{C}_{\lambda_0}$ is compact and there exists a bounded open set $\mathscr{O}\subset\mathscr{E}$ such that $\mathscr{C}_{\lambda_0}\subset\mathscr{O}$, $\partial\mathscr{O}\cap\mathscr{S}=\varnothing$, $\mathrm{Cl}((\mathscr{O}_R))\cap\operatorname{ess\,sp} L=\varnothing$, the only trivial solutions contained in $\mathscr{O}$ are points $(\lambda,0)$ where $|\lambda-\lambda_0|<\varepsilon$ for some $\varepsilon<\varepsilon_0=\operatorname{dist}(\lambda_0,\operatorname{sp} L\backslash\lambda_0)$, and $\operatorname{dist}(\partial\mathscr{O},\{\operatorname{sp} L\times\{0\}\})\geqq 2\varepsilon_1$ for some positive ε_1.*

REMARK. The theorem below will show that the hypotheses of the preceding lemma imply that λ_0 is an eigenvalue of even multiplicity.

THEOREM 1. *Let λ_0 be an isolated eigenvalue of L having odd multiplicity. Then*

(i) *$\mathscr{C}_{\lambda_0}$ is unbounded, or*

(ii) *$\mathscr{C}_{\lambda_0}$ is bounded and $\mathrm{Cl}((\mathscr{C}_{\lambda_0})_R)\cap\operatorname{ess\,sp} L\neq\varnothing$, or*

(iii) *$\mathscr{C}_{\lambda_0}$ is compact, $\mathrm{Cl}((\mathscr{C}_{\lambda_0})_R)\cap\operatorname{ess\,sp} L=\varnothing$, and $\mathscr{C}_{\lambda_0}$ contains trivial solutions other than $(\lambda_0,0)$.*

PROOF. Let us define $\Phi(\lambda,u)=Lu-\lambda u-H(\lambda,u)$. In general, degree theory cannot be applied to such an operator. Under the hypothesis on L we will show how Φ can be replaced by a compact perturbation of the identity, thus allowing the use of degree theory.

Assume that none of (i), (ii), and (iii) occurs. Then by Lemma 1 we find a bounded open set $\mathscr{O}$, $\varepsilon>0$, and $\varepsilon_1>0$, such that $\mathscr{C}_{\lambda_0}\subset\mathscr{O}$, $\mathrm{Cl}((\mathscr{O}_R))\cap\operatorname{ess\,sp} L=\varnothing$, $\partial\mathscr{O}\cap\mathscr{S}=\varnothing$, $\operatorname{dist}(\partial\mathscr{O},\{\operatorname{sp} L\times\{0\}\})\geqq 2\varepsilon_1$, and the only trivial solutions to (1) in $\mathscr{O}$ are points $(\lambda,0)$ satisfying $|\lambda-\lambda_0|<\varepsilon<\varepsilon_0$, where $\varepsilon_0=\operatorname{dist}(\lambda_0,\operatorname{sp} L\backslash\{\lambda_0\})$.

Select a neighborhood N of $\operatorname{ess\,sp} L$ which contains $\mathrm{Cl}((\mathscr{O}_R))$ in its exterior, and let $\mu_0\notin\mathrm{Cl}((\mathscr{O}_R))$ be in the resolvent set. Let $\mathscr{H}'$ denote the maximal closed subspace for which $L\mathscr{H}'\subseteq\mathscr{H}'$ and $\operatorname{sp} L|\mathscr{H}'=\operatorname{sp} L\cap N$, and let P be the projector onto $\mathscr{H}'$. Define the linear operator L_0 by

$$L_0=(L-\mu_0 I)(I-P).$$

L_0 is clearly compact. Furthermore, $\lambda\notin N$ is an eigenvalue of L having multiplicity m if and only if $\lambda-\mu_0$ is an eigenvalue of L_0 having multiplicity m. For $\lambda\notin\{\mu_0\}\cup N$ we define

$$G_\lambda=(\lambda-\mu_0)^{-1}[L_0+(I-P)(-H(\lambda,u))]+(\lambda-L)^{-1}P(-H(\lambda,u)).$$

From the definition of P it follows that (1) is equivalent to

$$u = G_\lambda u \tag{2}$$

for λ in a neighborhood of $\mathrm{Cl}((\mathcal{O}_R))$. The linear part of G_λ is compact and the linear part of G_{λ_0} has the eigenvalue 1 with multiplicity m_0 if and only if L has the eigenvalue λ_0 with multiplicity m_0. The nonlinear part of G_λ is also compact and in norm is $o(\|u\|)$ for small u.

(2) is the form necessary for the use of Leray-Schauder degree theory. Applying this theory as Rabinowitz [2] did shows that one of (i), (ii), or (iii) must occur.

REMARK. If the multiplicity of λ_0 is odd, Theorem 1 guarantees that λ_0 is a bifurcation point with a continuous branch $\mathscr{C}_{\lambda_0}$.

COROLLARY 1. *Let λ_0 be an isolated eigenvalue of L of finite multiplicity which is a bifurcation point with continuous branch λ_0. Then*

(i)′ *$\mathscr{C}_{\lambda_0}$ is unbounded, or*

(ii)′ *$\mathscr{C}_{\lambda_0}$ is bounded and $\mathrm{Cl}((\mathscr{C}_{\lambda_0})_R) \cap \mathrm{ess\ sp}\, L \neq \varnothing$, or*

(iii)′ *$\mathscr{C}_{\lambda_0}$ is compact, $\mathrm{Cl}((\mathscr{C}_{\lambda_0})_R) \cap \mathrm{sp}\, L = \{\lambda_0, \lambda_1, \cdots, \lambda_n\}$ and the sum of the multiplicities of the eigenvalues $\lambda_0, \lambda_1, \cdots, \lambda_n$ is even.*

We now consider

$$Lu = \lambda K u + H(\lambda, u), \tag{3}$$

where K is positive definite and bounded and L, H are as above.

COROLLARY 2. *Let R be the positive square root of K. Let λ_0 be an isolated eigenvalue of $R^{-1}LR^{-1}$ of finite multiplicity which is a bifurcation point of* (3) *with a continuous branch $\mathscr{D}_{\lambda_0}$. Then*

(i) *$\mathscr{D}_{\lambda_0}$ is unbounded, or*

(ii) *$\mathscr{D}_{\lambda_0}$ is bounded and $\mathrm{Cl}((\mathscr{D}_{\lambda_0})_R) \cap \mathrm{ess\ sp}(R^{-1}LR^{-1}) \neq \varnothing$, or*

(iii) *$\mathscr{D}_{\lambda_0}$ is compact, $\mathrm{Cl}((\mathscr{D}_{\lambda_0})_R) \cap \mathrm{sp}(R^{-1}LR^{-1}) = \{\lambda_0, \lambda_1, \cdots, \lambda_n\}$ and the sum of the multiplicities of the eigenvalues $\lambda_0, \lambda_1, \cdots, \lambda_n$ (of $R^{-1}LR^{-1}$) is even.*

If the multiplicity of λ_0 is odd, then $(\lambda_0, 0)$ is a bifurcation point possessing a continuous branch.

3. **General operators.** We now generalize by considering a real Banach space $\mathscr{B}$ and linear operators $T: \mathscr{B} \to \mathscr{B}$. The equation being studied is

$$Tu = \lambda u + H(\lambda, u) \tag{4}$$

with H as before.

THEOREM 2. *Suppose λ_0 is an isolated eigenvalue of T of odd multiplicity and*

(a) *to every closed interval $\sigma \subset R\backslash$ess sp T containing λ_0 there is a compact projector Q_σ that commutes with T, and λ_0 is an isolated eigenvalue of $T|Q_\sigma\mathscr{B}$ of odd multiplicity,*

(b) *the restriction of $T-\lambda I$ to $(I-Q_\sigma)\mathscr{B}$ is invertible for $\lambda \in \sigma$.*

Then $(\lambda_0, 0)$ is a bifurcation point possessing a continuous branch $\mathscr{C}_{\lambda_0}$ such that

(i) *$\mathscr{C}_{\lambda_0}$ is unbounded, or*

(ii) *$\mathscr{C}_{\lambda_0}$ is bounded and $\mathrm{Cl}((\mathscr{C}_{\lambda_0})_R) \cap \text{ess sp } T \neq \varnothing$, or*

(iii) *$\mathscr{C}_{\lambda_0}$ is compact, $(\mathscr{C}_{\lambda_0})_R \cap \text{sp } T = \{\lambda_0, \lambda_1, \cdots, \lambda_n\}$ and the sum of the multiplicities of the eigenvalues $\lambda_0, \lambda_1, \cdots, \lambda_n$ is even.*

PROOF. The proof is similar to that of Theorem 1.

COROLLARY 3. *Suppose λ_0 is an isolated eigenvalue of T of odd multiplicity and for every closed interval $\sigma \subset R\backslash$ess sp T containing λ_0, T can be uniformly approximated by operators T_ε which are of the type treated in Theorem 2 and such that $\text{sp } T_\varepsilon \cap \sigma = \text{sp } T \cap \sigma$ up to multiplicity of eigenvalues. Then the results of Theorem 2 hold for T and $\mathscr{C}_{\lambda_0}$.*

Our work necessitates the use of a complexification of $\mathscr{B}$ which is denoted by $\hat{\mathscr{B}} = \mathscr{B} \times \mathscr{B}$. The general element of $\hat{\mathscr{B}}$ is

$$(x, y) = x + iy \quad \text{and} \quad \|(x, y)\|_{\hat{\mathscr{B}}} = (\|x\|^2 + \|y\|^2)^{1/2},$$

where $\|\cdot\|$ is the norm in $\mathscr{B}$. For any linear $T: \mathscr{B} \to \mathscr{B}$, $\hat{T}: \hat{\mathscr{B}} \to \hat{\mathscr{B}}$ is its unique linear extension to $\hat{\mathscr{B}}$.

THEOREM 3. *Let T be a bounded linear operator and σ be a compact subset of $R\backslash$ess sp $\hat{T}$. Then there is a bounded projector Q_σ that commutes with T such that the restriction of $T-\lambda I$ to $(I-Q_\sigma)\mathscr{B}$ is invertible for $\lambda \in \sigma$ and $Q_\sigma\mathscr{B}$ is the span of the principal manifolds belonging to eigenvalues of T in σ.*

PROOF. The first step is to go to the complexifications $\hat{T}$ and $\hat{\mathscr{B}}$. A decomposition theorem [3] is applicable to this complex case. From this complex decomposition, we can derive suitable real projections from $\mathscr{B}$ into $\mathscr{B}$ and their corresponding subspaces in $\mathscr{B}$.

REMARK. It follows from this theorem that Theorem 2 holds for all bounded linear operators T on $\mathscr{B}$ for which $R \cap \text{ess sp } \hat{T} = \text{ess sp } T$. In particular this is true if T is compact, or if $\mathscr{B}$ is a Hilbert space and T is selfadjoint.

References

1. M. A. Krasnosel′skiĭ, *Topological methods in the theory of nonlinear integral equations*, GITTL, Moscow, 1956; English transl., Macmillan, New York, 1964. MR **20** #3464; **28** #2414.

2. P. H. Rabinowitz, *Some aspects of nonlinear eigenvalue problems*, Rocky Mountain J. Math. **3** (1973), no. 2, 161–202.

3. F. Riesz and B. Sz.-Nagy, *Leçons d'analyse fonctionnelle*, 2nd ed., Akad. Kiadó, Budapest, 1953; English transl., *Functional analysis*, Ungar, New York, 1971. MR **15**, 132.

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