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GENERALIZED GRADIENT FIELDS AND ELECTRICAL CIRCUITS[1]

BY ALCIDES LINS NETO

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1. **Introduction.** In *On the mathematical foundations of electrical circuit theory*, Smale [**S.1**] proposes the following two problems.

Problem 1.1. What can one say about the dynamical systems which are gradient systems with respect to a nondegenerate indefinite metric, say on a compact manifold.

Problem 1.2. Can one always regularize the equations of (1.6) [**S.1**], by adding arbitrarily small inductors and capacitors to the circuit appropriately? How? By regularizing we mean obtaining new equations which have the property $\pi:\sum\to\mathscr{L}\times\mathscr{C}'$ is a local diffeomorphism.

Furthermore he makes the following conjecture

CONJECTURE 1.3. Suppose $X=\operatorname{grad}(\omega)$ is the gradient of a closed 1-form with respect to a Riemannian metric on a compact manifold M. Suppose further that ω is not cohomologous to zero and that X is well behaved in the sense that it satisfies the conditions of [**S.2**, (2.2)]. Then X has a closed orbit, not a point, which is aymptotically stable (i.e. a sink).

In this work we give a counterexample to this conjecture. Furthermore we reformulate it, solving the new version in the case $\dim M=2$. For Problem 1.1 we obtain generic properties for the generalized gradient fields as in the Kupka-Smale theorem. Moreover we characterize structural stability for these types of vector fields in the case M is compact, orientable, and dim=2. For Problem 1.2 we give a counterexample in the general case and solve the problem imposing conditions on the resistors of the circuit.

Before we state the theorems we need some definitions and notations. M will be a C^∞ manifold (with or without boundary), $TM\oplus TM=\{(p, v, w)\mid p\in M, v, w\in TM_p\}$.

DEFINITION 1.4. A metric C^r on M is a C^r map $\mu: TM\oplus TM\to \boldsymbol{R}$, such that for each $p\in M$, the map $\mu_p: TM_p\times TM_p\to\boldsymbol{R}$ given by $\mu_p(v, w)=\mu(p, v, w)$ is bilinear symmetric. We say that μ is a nondegenerate metric on M if for each $p\in M$, μ_p is a nondegenerate bilinear form on TM_p.

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[1] This is a resume of my dissertation, developed at IMPA under the guidance of Professor Cesar Camacho.

NOTATIONS 1.5. $\Lambda^{1,r}(M)$ is the set of C^r, 1-forms on M and $\mathfrak{X}^r(M)$ is the set of C^r vector fields on M. Here we will consider $\Lambda^{1,r}(M)$ and $\mathfrak{X}^r(M)$ endowed with the Whitney C^r topology. $\mathscr{F}^r(M)$ will denote the set of closed C^r 1-forms on M and $\xi^r(M)$ the set of exact C^r 1-forms on M.

We remark that $\xi^r(M) \subset \mathscr{F}^r(M)$ are closed linear subspaces of $\Lambda^{1,r}(M)$.

DEFINITION 1.6. Let μ be a nondegenerate metric on M and $\omega \in \Lambda^{1,r}(M)$. We say that $X \in \mathfrak{X}^r(M)$ is the gradient of ω with respect to μ if for each $p \in M$ and $v \in TM_p$ we have

$$\mu_p(X(p), v) = \omega_p(v). \tag{1.6.1}$$

In this case we denote $X = \operatorname{grad}_\mu(\omega)$. It is not difficult to show that the equality (1.6.1) defines a unique C^r vector field on M and that the map $\operatorname{grad}_\mu : \Lambda^{1,r}(M) \to \mathfrak{X}^r(M)$ is an isomorphism of topological vector spaces.

NOTATIONS 1.7. In this section we fix a nondegenerate metric μ on M.

1.7.1. $\mathscr{F}^r_\mu(M) = \operatorname{grad}_\mu(\mathscr{F}^r(M))$ is the set of vector fields $X \in \mathfrak{X}^r(M)$, such that $X = \operatorname{grad}_\mu(\omega)$, $\omega \in \mathscr{F}^r(M)$.

1.7.2. $\xi^r_\mu(M) = \operatorname{grad}_\mu(\xi^r(M))$.

1.7.3. Let $M - S(\mathfrak{X}^r(M))$ be the set of Morse-Smale vector fields on M (cf. [**P.3**]). If $\mathscr{N} \subset \mathfrak{X}^r(M)$, then $\mathscr{N} \cap M - S(\mathfrak{X}^r(M))$ is denoted by $M - S(\mathscr{N})$.

1.7.4. Let $\mathscr{G}_{123}(\mathfrak{X}^r(M))$ be the set of Kupka-Smale vector fields on M (cf. [**P.1**]). If $\mathscr{N} \subset \mathfrak{X}^r(M)$, then $\mathscr{N} \cap \mathscr{G}_{123}(\mathfrak{X}^r(M))$ is denoted by $\mathscr{G}_{123}(\mathscr{N})$.

1.7.5. We say that a closed C^1 curve $\gamma : [a, b] \to M$ is transversal to $\omega \in \mathscr{F}^r(M)$ if for each $t \in [a, b]$, $\omega_{\gamma(t)}(\gamma'(t)) \neq 0$.

Now we can state the results concerning Conjecture 1.3 and Problems 1.1, 1.2.

PROPOSITION 1.8. *Let $\omega \in \mathscr{F}^r(M)$. Then there exists a Riemannian metric μ, such that* $\operatorname{grad}_\mu(\omega)$ *has a closed orbit which is an attractor if and only if ω admits a closed transversal.*

THEOREM A. *Let M be a C^∞ compact, orientable, 2-dimensional manifold, $\partial M = \varnothing$ and $M \neq S^2$. Then the set of closed 1-forms which admit a closed transversal is open and dense in $\mathscr{F}^r(M)$.*

SKETCH OF THE PROOF. We note that only density offers some difficulty. The idea of the proof is to show that $\omega \in \mathscr{F}^r(M^2)$ can be approached by a 1-form $\tilde{\omega} \in \mathscr{F}^r(M)$, such that there is a leaf γ of the foliation induced by $\tilde{\omega}$ on M^2 which is nontrivially recurrent. Thus $\tilde{\omega}$ admits a closed transversal. We note that the perturbation of ω cannot be made locally, and in fact it is made in a tubular neighborhood of a closed curve on M^2 which intersects some leaf of ω transversally in a unique point.

We note that it is not difficult to construct an open set of closed 1-forms on $S^2\times S^1$ which do not admit a closed transversal (note that $\xi^r(S^2\times S^1)\neq \mathscr{F}^r(S^2\times S^1)$).

A natural question is: For which manifolds M is the set of closed 1-forms which admit a closed transversal dense? We conjecture that a sufficient condition for dim $M\geqq 3$ is $H_1(M; \boldsymbol{R})\neq 0$ and $\pi_{n-1}(M)=0$.

For Problem 1.1 we have the following results:

THEOREM B. *Let M be a C^∞ manifold. Then $\mathscr{G}_{123}(\mathscr{F}^r_\mu(M))$ $[\mathscr{G}_{123}(\xi^r_\mu(M))]$ is residual in $\mathscr{F}^r_\mu(M)$ $[\xi^r_\mu(M)]$ (μ is a fixed nondegenerate metric on M).*

SKETCH OF THE PROOF. The idea is to use a technique introduced by Abraham in [**A-R**, §§31, 32, 33]. We note that such techniques cannot be applied crudely to the problem when we are restricted to $\mathscr{F}^r_\mu(M)$. The following example shows the main difficulty.

Let $Q=\{(t,x)\in \boldsymbol{R}^2 | (t)\leqq 1, |x|\leqq 1\}$ and μ be the metric on $\boldsymbol{R}^2$ whose quadratic form is $2\,dt\,dx$. Given $f:\boldsymbol{R}^2\to\boldsymbol{R}$ we have $\text{grad}_\mu(df)=(\partial f/\partial y, \partial f/\partial x)$. Consider the Banach space

$$N = \{\eta\in\xi^r(\boldsymbol{R}^2) \mid \eta(p) = 0 \text{ in } p\notin Q\}.$$

Let $f:\boldsymbol{R}^2\to\boldsymbol{R}$ be given by $f(t,x)=x$ and $\omega=df$. Then $\text{grad}_\mu(\omega)=(1,0)$. Let N_1 be the open set of N, defined by

$$N_1 = \{\eta\in N \mid \text{grad}_\mu(\omega+\eta) = (Y_1, Y_2) \text{ with } Y_1(p) > \tfrac{1}{2}\forall\, p\in Q\}.$$

Let $\Sigma_1=\{(t,x)\in Q | t=1\}$ and $F:N_1\to\Sigma_1$ be defined by $F(\eta)=$the point where the orbit of $\text{grad}_\mu(\omega+\eta)$, by the point $(-1,0)$, intersects Σ_1. Then F is C^1. To use Abraham's techniques it is essential that F be a submersion. In this example F is not a submersion at $\eta=0$.

This difficulty is overcome by restricting the analysis to an open and dense subset τ of $\mathscr{F}^r_\mu(M)$. Then we show that $\mathscr{G}_{123}(\mathscr{F}^r_\mu(M))$ is residual in τ.

THEOREM C. *Let M be a C^∞, compact manifold with $\partial M=\varnothing$. Let μ be a Riemannian metric on M. Then $M-S(\xi^r_\mu(M))$ is dense in $\xi^r_\mu(M)$.*

Since the Morse-Smale vector fields are structurally stable (cf. [**P-S**]), it follows from Theorem C and minor arguments that the set of structurally stable vector fields in $\xi^r_\mu(M)$ is $M-S(\xi^r_\mu(M))$.

Smale [**S.3**] proves a weaker form of Theorem C. There he perturbs both the metric μ and the 1-form $\omega=df$.

THEOREM D. *Let M be a C^∞ compact, orientable, 2-dimensional manifold with $\partial M=\varnothing$. Let μ be a nondegenerate metric on M. Then $M-S(\mathscr{F}^r_\mu(M))$ $[M-S(\xi^r_\mu(M))]$ is dense in $\mathscr{F}^r_\mu(M)$ $[\xi^r_\mu(M)]$.*

The idea of the proof is to use the techniques developed by Peixoto in **[P.2]**.

In §2 we give an example which shows that the answer to Problem 1.2, in the general case, is no. However we have the following result:

THEOREM E. *Let G be a circuit which satisfies*:

(a) *The projection* $i_L \times v_C: K \to \mathscr{L} \times \mathscr{C}'$ (*cf.* [**S.1**, p. 4]) *is surjective.*

(b) *If* ρ *is a resistor of G and* Λ_ρ *its characteristic curve, then* Λ_ρ *is the graph of a function* (A) $i_\rho = f(v_\rho)$ *or* (B) $v_\rho = f(i_\rho)$, *where* $f: \boldsymbol{R} \to \boldsymbol{R}$ *is a smooth function.*

Then G is regularizable in the sense of [**S.1**, (3.3)]. *Furthermore if* G_1 *is the new circuit and* $\pi_1: \Sigma_1 \to \mathscr{L}_1 \times \mathscr{C}_1'$, *then* π_1 *is a* C^1 *diffeomorphism.*

The idea of the proof is to insert inductors in series with some of the resistors of type A and capacitors in parallel with some of the resistors of type B.

2. Examples.

EXAMPLE 2.1. This is a counterexample to Conjecture 1.3. Let $M = T^n = \boldsymbol{R}^n/(2\pi\boldsymbol{Z})^n$. We have the natural covering map $\boldsymbol{R}^n \to^p T^n$, such that p identifies points $(x_1, \cdots, x_n)$, $(x_1', \cdots, x_n') \in \boldsymbol{R}^n$ where $(x_i - x_i')/2\pi \in \boldsymbol{Z}$. Let μ be the Riemannian metric on T^n induced by the euclidean metric $\tilde{\mu}$ of $\boldsymbol{R}^n$. Let

$$\tilde{\omega}(x_1, \cdots, x_n) = \sum_{i=1}^{n} (1 - 2 \sin x_i)\, dx_i.$$

Then $\tilde{\omega}$ is closed and there exists a closed 1-form ω on T^n such that $\tilde{\omega} = p^*(\omega)$. It is not difficult to see that ω is not exact. Let $X = \operatorname{grad}_\mu(\omega)$. Then X is C^∞ and $X = (X_1, \cdots, X_n)$ (in coordinates) where X_i is a Morse-Smale vector field on S^1 with two singularities, a sink and a source. Therefore X is a Morse-Smale vector field on T^n without closed orbits.

EXAMPLE 2.2. Let ρ be a resistance such that its characteristic curve $\Lambda_\rho \subset R^2$ has tangents in all directions.

ASSERTION. *Let G be a circuit such that* ρ *is the unique resistor of G. Then G is not regular.* (We suppose obviously that ρ is not the unique element of G.) Write the currents and voltages of G as

$$(i, v) = (x, y, z, x', y', z'),$$

where $(x, x') \in \mathscr{L} \times \mathscr{L}'$, $(y, y') \in \mathscr{C} \times \mathscr{C}'$ and $(z, z') \in \mathscr{R} \times \mathscr{R}' \simeq \boldsymbol{R}^2$ (for the notations see [**S.1**]). Let $\pi': K \to \mathscr{R} \times \mathscr{R}'$, $\Sigma \subset K$ and $\pi: \Sigma \to \mathscr{L} \times \mathscr{C}'$ be as in [**S.1**]. Then π' is surjective (because Kirchhoff laws do not impose restrictions in $\mathscr{R} \times \mathscr{R}'$) and $\Sigma = (\pi')^{-1}(\Lambda_\rho)$ is a submanifold of K. If $p = (x, y, z, x', y', z') \in \Sigma$ we have $T\Sigma_p = \{(\dot{x}, \dot{y}, \dot{z}, \dot{x}', \dot{y}', \dot{z}') = \dot{p} \mid \dot{p} \in K$ and

$(\dot z, \dot z') \in (T\Lambda_\rho)_{(z,z')}\}$ and $D\pi_p(\dot p)=(\dot x, \dot y') \in \mathscr{L}\times\mathscr{C}'$. If there exists $p \in \Sigma$, such that $D\pi_p$ is surjective, then the projection $i_L \times v_c : K \to \mathscr{L}\times\mathscr{C}'$ is surjective. Now

$$\dim(\ker(i_L \times v_c)) = \dim(K) - \dim(\mathscr{L} \times \mathscr{C}') = 1.$$

Let $\dot p=(\dot x, \dot y, \dot z, \dot x', \dot y', \dot z') \in \ker(i_L \times v_c)$, $\dot p \neq 0$. By the hypothesis, there exists $(z, z') \in \Lambda_\rho$, such that $(\dot z, \dot z') \in (T\Lambda_\rho)_{(z,z')}$, therefore $\dot p \in T\Sigma_p$ where $\pi'(p)=(z, z')$ and $D\pi_p(\dot p)=0$, $\dot p\neq 0$. This shows that π is not a local diffeomorphism at p.

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Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brasil