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RADICAL EMBEDDING, GENUS, AND TOROIDAL DERIVATIONS OF NILPOTENT ASSOCIATIVE ALGEBRAS

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ABSTRACT. The author continues to discuss this problem: given a nonzero nilpotent finite-dimensional associative algebra N over the perfect field k, describe the set of unital associative k-algebras A satisfying the equation rad $A=N$, together with the "nowhere triviality" condition $\mathrm{Ann}_A N \subset N$. In this paper the Lie homomorphism $\delta: S_{\mathrm{Lie}} \to \mathrm{Der}_k N$ induced by bracketing (where A has Wedderburn decomposition as semidirect sum $S+N$) is studied as follows: (i) the kernel and image of δ are computed; (ii) conditioning the derivation algebra $\mathrm{Der}_k N$ conditions the semisimple S; (iii) for instance, $\mathrm{Der}_k N$ solvable implies that S is a direct sum of fields; (iv) those tori in $\mathrm{Der}_k N$ of the form δS are characterized in terms of their 0-weightspace in N.

1. **Introduction.** For previous discussions, see Hall [2] and Flanigan [1]. Throughout, N is a given finite-dimensional nilpotent k-algebra with k perfect. We seek those semisimple k-algebras S which satisfy the following conditions.

(1.1) DEFINITION [1]. *N accepts S as a nowhere trivial Wedderburn factor* if there is a unital associative k-algebra S A such that (i) $A \simeq N+S$ (Wedderburn decomposition), and (ii) $S \cap \mathrm{Ann}_A N=(0)$.

Note that (ii) forces A to be finite dimensional, and that $N \neq (0)$ implies $S \neq (0)$. In [1] we examined candidates S for acceptance by considering such invariants of N as its quotients N/N^i and its graded form gr N. Now we utilize the Lie algebra $\mathrm{Der}_k N$ of k-algebra derivations $N \to N$ by noting that, if N accepts S as in (1.1), then there is a Lie homomorphism

$$\delta: S_{\mathrm{Lie}} \to \mathrm{Der}_k N \tag{1.2}$$

with $\delta(b)x=[b, x]=bx-xb$ for all x in N, b in S, and with the products taken in A.

We are particularly interested in those S which are direct sums of fields. *Reason*: the center of *every* semisimple algebra accepted by N would be of this type. These direct sums of fields are determined by the

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"genus" of N (§3). Thus, genus$(N)=0$ means that the only S, commutative or not, accepted by N is essentially that obtained by the well-known process of adjoining a unity to N. In §4 we bound genus(N) in terms of the dimension of maximal tori in $\mathrm{Der}_k N$ and from this draw consequences for S. The family of examples in §5 shows that this upper bound on genus(N) may or may not be attained for a given N, and if not, it is because there exists an abelian S_{Lie} and Lie homomorphism $S_{\mathrm{Lie}}\to \mathrm{Der}_k N$ which is *not* induced by bracketing (see (1.2)) in an associative $A=N+S$. Finally, we identify those tori ("Peirce tori") in $\mathrm{Der}_k N$ which *are* of the form $\delta(S_{\mathrm{Lie}})$ in terms of the associative algebra structure of their 0-weightspaces in N (§6). The Peirce tori are those whose weight-space decomposition of N is essentially a Peirce (idempotent) decomposition in the classical sense.

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2. **The Lie homomorphism δ.** It often makes good sense to specify that (i) N is indecomposable (into two-sided ideals) as a k-algebra, and that (ii) the semisimple k-algebra S is *split over* k, that is, S is an ideal direct sum of total matrix algebras $M(r_\alpha, k)$ of rank r_α with all entries in k. This latter is always the case if k is algebraically closed.

(2.1) LEMMA. *Let the nonzero indecomposable nilpotent k-algebra N accept the split semisimple* $S=\bigoplus_\alpha M(r_\alpha, k)$, *with* $\alpha=1,\cdots,s$, *as a nowhere trivial Wedderburn factor. Then the map* δ *of* (1.2) *satisfies*

(i) $\ker(\delta)$ *is the one-dimensional Lie ideal* $k\cdot 1=k\cdot 1_S$;

(ii) *if* $\mathrm{char}(k)$ *divides none of the ranks* r_α, *then the image of* δ *is isomorphic with the Lie algebra*

$$\Big(\bigoplus_\alpha sl(r_\alpha, k)\Big)\oplus\Big(\Big(\bigoplus_\alpha k\cdot e_\alpha\Big)\Big/k\cdot 1\Big)\qquad (\alpha=1,\cdots,s),$$

where e_α *is the unity element of* $M(r_\alpha, k)$;

(iii) *S is a direct sum of copies of k, that is, all* $r_\alpha=1$ *if and only if the image of* δ *is a torus in* $\mathrm{Der}_k N$.

(2.2) COMMENTS. (a) If $s\geqq 2$, then $\ker(\delta)$ is therefore a proper subalgebra of the center of S_{Lie};

(b) The Lie algebra $sl(r_\alpha, k)$ consists of the matrices with trace zero;

(c) $k\cdot e_\alpha$ is the subalgebra of scalar matrices in $M(r_\alpha, k)$, and $1=1_S=e_1+\cdots+e_s$;

(d) A *torus* is an abelian linear Lie algebra consisting of semisimple operators;

(e) The Lemma follows from elementary considerations of two-sided matrix actions on N. The proof does not require nilpotence of N, but only that N be an ideal in $A=S+N$. The Lemma is false, however, if N is itself decomposable as a direct sum of two-sided ideals.

(2.3) *Question.* If N accepts a maximal (see [**1**]) split S, does the map δ always send the center of S into the *solvable radical* of $\mathrm{Der}_k N$? An affirmative answer would yield a much more severe constraint on S. This would be reflected in statement (iii) of Theorem (4.1), where the integer $t(N)$ could then be replaced by a smaller and better understood number, the dimension of a maximal torus in the solvable radical of $\mathrm{Der}_k N$.

3. Genus(N) **and** $t(N)$. The genus will provide a measure of the "fineness" of the Peirce decompositions which N admits.

(3.1) DEFINITION. If N is a nonzero nilpotent k-algebra, then $\mathrm{genus}(N)=\max_S(\dim_k S)-1$, where $S=ke_1\oplus\cdots\oplus ke_s$ is a direct sum of s copies of the field k accepted by N as a nowhere trivial Wedderburn factor.

Thus $\mathrm{genus}(N)\geqq 0$ and, if $N=I_1\oplus\cdots\oplus I_q$ is a decomposition into nonzero two-sided ideals, then one readily checks that $\mathrm{genus}(N)=q-1+\sum_i \mathrm{genus}(I_i)$, and that this is $\geqq 1$ if $q\geqq 2$.

(3.2) EXAMPLE. Let N be the nilpotent algebra of all strictly upper triangular n by n matrices over k. Then $\mathrm{genus}(N)=n-1$. See [**1**, (2.3)].

(3.3) EXAMPLE. Let N be the truncated polynomial ideal generated by linearly independent (over k) noncommuting elements $x_1, \cdots, x_m$ such that every monomial of degree $\geqq \nu+1$ reduces to zero, so that $N^\nu\neq(0)$ but $N^{\nu+1}=(0)$. If $\nu\geqq 2$, then N is indecomposable and $\mathrm{genus}(N)=0$ independent of m. See [**1**, (2.4)].

The following invariant of N was introduced by Leger and Luks [**3**, §1] to study nilpotent Lie algebras.

(3.4) DEFINITION. $t(N)$ is the dimension of a maximal torus in the derivation algebra $\mathrm{Der}_k N$.

Thus, if $\mathrm{Der}_k N$ is nilpotent, then $t(N)=0$.

4. **Results on** S. These follow from Lemma (2.1) and the basic structure of algebraic Lie algebras.

(4.1) THEOREM. *Let the nonzero indecomposable nilpotent k-algebra N accept as nowhere trivial Wedderburn factor the split semisimple $S=\bigoplus_\alpha M(r_\alpha, k)$, with $\alpha=1, \cdots, s$. Then*

(i) *if* char $k=0$ *and a Levi factor of* $\mathrm{Der}_k N$ *has no nonzero subalgebras $sl(n, k)$, then S is a direct sum of copies of k;*

(ii) *if* $\mathrm{Der}_k N$ *is solvable, then S is a direct sum of copies of k;*

(iii) $(\sum_\alpha r_\alpha)-1\leqq \mathrm{genus}(N)\leqq t(N)$;

(iv) *in particular, if* $\mathrm{Der}_k N$ *is nilpotent, then* $\mathrm{genus}(N)=0$, *that is,* $S=k$.

5. **An illustration.** This family of algebras will provide counterexamples to the converses of certain assertions in (2.1) and (4.1). Let char $k\neq 2$ and, for each τ in k, let N_τ be the 3-dimensional k-algebra with basis x, y, z and multiplication $xy=z, yx=\tau z$, and all other products of basis elements zero. Note that $(N_\tau)^3=(0)$, so that N_τ is associative, nilpotent, and indecomposable.

The following assertions about N_τ are easily verified.

(5.1) N_0 accepts $S=ke_1\oplus ke_2\oplus ke_3$ (cf. 3 by 3 upper triangular matrices). Genus$(N_0)=2$. Also $t(N_0)=2$.

(5.2) For $\tau\neq 0$, genus$(N_\tau)=0$, but again $t(N_\tau)=2$.

(5.3) All $\mathrm{Der}_k\, N_\tau$ are solvable nonnilpotent with 2-dimensional maximal torus.

(5.4) The maximal tori in all $\mathrm{Der}_k\, N_\tau$ are isomorphic, and all modules N_τ are equivalent.

(5.5) *Moral.* The structure of $\mathrm{Der}_k\, N$ and its natural representation on N (as discussed so far) are not sufficient to decide genus(N). The conditions we give in §6 that a torus in $\mathrm{Der}_k\, N$ be of the form $\delta(S)$ as in (1.2) must necessarily involve the associative product in N.

6. **Peirce tori and direct sums of fields.** We characterize in terms of $\mathrm{Der}_k\, N$ the direct sums of fields accepted by N. Here "eigenvalues" and "weights" refer to the natural representation of $\mathrm{Der}_k\, N$ on N.

(6.1) DEFINITION. Let N be an indecomposable nilpotent k-algebra. The torus T in $\mathrm{Der}_k\, N$ is a *Peirce torus* if either $T=(0)$ or T has a spanning set $\varepsilon_1, \cdots, \varepsilon_m$ with $m\geqq 2$ satisfying these four conditions:

(a) $\varepsilon_1+\cdots+\varepsilon_m=0$, but any $m-1$ of the ε_i furnish a k-basis for T;

(b) the set of eigenvalues for each ε_i is either $\{0, 1\}$, $\{0, -1\}$, or $\{0, 1, -1\}$;

(c) each nonzero weight of T is of the form λ_{ij}, defined by $\lambda_{ij}(\varepsilon_i)=1$, $\lambda_{ij}(\varepsilon_j)=-1$, $\lambda_{ij}(\varepsilon_h)=0$ for h, i, j distinct;

(d) the 0-weightspace W_0 in N decomposes as k-algebra into a direct sum $\bigoplus_i W_i$ of two-sided (possibly zero) ideals, $i=1, \cdots, m$, satisfying (here W_{ij} is the λ_{ij}-weightspace) for distinct h, i, j,

$$W_{ij}W_{ji}\subset W_i, \qquad W_iW_{hj}=(0), \qquad W_{hj}W_i=(0).$$

A Peirce torus yields a standard Peirce decomposition $N=\bigoplus_{i,j} e_iNe_j$ with respect to orthogonal $e_1, \cdots, e_m$ via the definitions $e_iNe_i=W_i$, $e_iNe_j=W_{ij}$ for distinct i, j.

(6.2) THEOREM. *The indecomposable nilpotent k-algebra N accepts* $S=ke_1\oplus\cdots\oplus ke_s$ *as nowhere trivial Wedderburn factor if and only if* $\mathrm{Der}_k\, N$ *contains a Peirce torus of dimension* $s-1$.

References

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