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A CONVERGENT FAMILY OF DIFFUSION PROCESSES WHOSE DIFFUSION COEFFICENTS DIVERGE

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1. **Introduction.** The problem of deriving suitable hypotheses from which one can conclude the weak convergence of a family of diffusion processes $x_n(t, \omega)$ to a limiting diffusion process $x(t, \omega)$ as $n \to \infty$ has attracted much attention in recent years. One popular approach is to study the asymptotic behavior of the diffusion coefficients. Specifically let us assume that $\{x_n(t, \omega), 1 \leq n < +\infty\}$ and $x(t, \omega)$ are one-dimensional, strong Markov processes with continuous paths and stationary transition probabilities. Assume further that the infinitesimal generators G_n and G of the corresponding semigroups

$$T_n(t)f(x) = E_x f(x_n(t, \omega)) \quad \text{and} \quad T(t)f(x) = E_x f(x(t, \omega))$$

are classical second order differential operators of the form:

$$\begin{aligned} G_n f(x) &= a_n(x) f''(x) + b_n(x) f'(x), \qquad a_n(x) > 0, \\ Gf(x) &= a(x) f''(x) + b(x) f'(x), \qquad a(x) > 0. \end{aligned} \tag{1}$$

Under sufficiently stringent hypotheses, Skorohod [**8**], Borovkov [**1**], Stroock-Varadhan [**9**], among others, have shown that a condition of the form

$$\lim_{n \to \infty} a_n(x) = a(x), \qquad \lim_{n \to \infty} b_n(x) = b(x) \tag{2}$$

is sufficient to conclude convergence of the semigroups, i.e.

$$\lim_{n \to \infty} T_n(t) f(x) = T(\ ) f(x) \tag{3}$$

for all f in a sufficiently large class of functions. It is known however that the infinitesimal generator G of the diffusion process $x(t, \omega)$ need not be a classical second order differential operator but instead can be one

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of Feller's generalized second order differential operators of the form

$$Gf(x) = D_m D_p^+ f(x), \tag{4}$$

where m and p are the speed and scale measures of the diffusion process $x(t, \omega)$. Of course every classical operator G can be put into the Feller form (4) via the recipe (see Mandl [**6**]):

$$\begin{aligned} dp(x) &= \exp(-B(x))\, dx, \\ dm(x) &= a(x)^{-1} \exp(B(x))\, dx, \\ B(x) &= \int^{x} b(y) \cdot a(y)^{-1}\, dy. \end{aligned} \tag{5}$$

The converse is not true. For example if $a(x)>0$ is continuous and $b(x)$ is locally integrable, then $p'(x)$ and $m'(x)$ exist everywhere and it is very easy to construct p and m for which these derivatives do not exist everywhere. These remarks suggest that it ought to be possible to have a family of diffusion processes $x_n(t, \omega)$ converging weakly as $n \to \infty$ to a limiting diffusion process $x(t, \omega)$ with the following properties: The infinitesimal generators G_n are classical operators of the form (1), but the infinitesimal generator G of the limit process is a generalized operator of the form (4) which is not a classical second order differential operator. In part 2 we give a simple, by no means artificial, example of exactly this kind of phenomenon from which we shall deduce the following interesting consequences:

THEOREM 1. *Any condition for the weak convergence of diffusion processes based on the asymptotic behavior of the diffusion coefficients is merely sufficient—it is not necessary.*

THEOREM 2. *The class of diffusion processes with stochastic integral representations of the form*

$$dx(t, \omega) = \sigma(x(t, \omega))\, dw(t, \omega) + b(x(t, \omega))\, dt$$

(here $w(t, \omega)$ is the Wiener process) is not closed with respect to weak convergence of stochastic processes.

2. **The example.** Let $x(t)$ (to simplify the notation we drop the ω) be the solution to the stochastic differential equation

$$dx(t) = dw(t) + b(x(t))\, dt \tag{6}$$

where $b(x)$ is in $L^1(-\infty, \infty)$ and $\int_{-\infty}^{\infty} b(x)\, dx = \alpha$. Let $x_n(t) = x(n^2 t)/n$. Then $x_n(t)$ satisfies the stochastic differential equation

$$dx_n(t) = dw(t) + b_n(x_n(t))\, dt \tag{7}$$

where $b_n(x) = nb(nx)$.

The following theorem is proved in the author's paper [7].

THEOREM 3. *If* $\alpha=0$ *then* $x_n(t)$ *converges weakly to the Wiener process as* $n\to\infty$.

If $\alpha\neq 0$ then $x_n(t)$ converges weakly to a diffusion process whose infinitesimal generator G has the Feller form $D_m D_p^+$, where $p(x)=c_1 x$, $x\geqq 0$, and $p(x)=c_2 x$ if $x\leqq 0$, $m(x)=2c_1^{-1}x$, $x\geqq 0$, and $m(x)=2c_2^{-1}x$ if $x\leqq 0$, and $c_2=c_1\exp(2\alpha)$—so $\alpha\neq 0$ implies that $c_1\neq c_2$ and hence $p'(0)$ and $m'(0)$ do not exist. Clearly the limit process depends on α.

REMARK. This is a far reaching generalization of a theorem of Gihman-Skorohod to be found in [3, p. 152].

It is obvious from (7) that $\lim_{n\to\infty}|b_n(0)|=+\infty$ if $b(0)\neq 0$, and, in fact, if $b(x)$ does not vanish at infinity, then $\lim_{n\to\infty} b_n(x)$ need not exist for any x. Indeed if $\alpha\neq 0$, there are no diffusion coefficients (in the classical sense) to which $a_n(x)\equiv\frac{1}{2}$ and $b_n(x)=nb(nx)$ can converge. This establishes the assertion of Theorem 1. It is equally clear that when $\alpha\neq 0$, $dx(t)$ does not possess a stochastic integral representation, for otherwise its infinitesimal generator would be a classical operator of the type defined at (1).

Our proof of Theorem 3 is based on the simple observation that

$$\lim_{n\to\infty} p_n(x) = p(x) \quad\text{and}\quad \lim_{n\to\infty} m_n(x) = m(x), \tag{8}$$

where m_n and p_n are the speed and scale measures of the $x_n(t)$ process. (The reader can easily check this for himself using (5).) From this fact we deduce the convergence of the resolvents, i.e.

$$\lim_{n\to\infty} \|(\lambda - G_n)^{-1}f - (\lambda - G)^{-1}f\| = 0 \tag{9}$$

for every $\lambda>0$ and every $f\in C_0(R)$, the bounded continuous functions vanishing at infinity.

From the Trotter-Kato theorem (see [10], [11]) we conclude immediately that

$$\lim_{n\to\infty}\ \sup_{0\leqq t\leqq z} |T_n(t)f - T(t)f| = 0 \quad \text{all } f\in C_0(R). \tag{10}$$

REMARK. This example can also be exploited to show that various conditions due to Trotter [10, Theorem 5.3], Chernoff [2], and Skorohod [8, Theorem 4.6] of the type

$$\lim_{n\to\infty} G_n f(x) = Gf(x) \tag{11}$$

are sufficient and not necessary. For another example of this phenomenon see a recent paper of Goldstein [4]. Our example also emphasizes the importance of Kurtz' work [5], who does obtain a *necessary* and sufficient condition for (10) to hold in terms of a different notion of convergence. We note, in conclusion, that while there is no generalization of the "Feller form" to higher-dimensional space the reformulation of the problem in terms of convergence of the resolvents suffers from no such defect.

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