SCHUR HOMOTOPY OF THE SIMPLEST GROUP

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Let G be a *perfect* group; that is, $G=[G, G]$, the commutator subgroup of G, or equivalently $H_1(G; Z)=0$. In [1], Quillen has shown that for any space X with $\pi_1 X$ perfect, there are a simply-connected space X^+ unique up to homotopy type and a map $j:X\to X^+$ such that $j_*:H_*(X; C)\to H_*(X^+; C)$ is an isomorphism for all coefficient groups C; moreover, any two maps inducing the same isomorphism $H_*(X; Z)\to H_*(X^+; Z)$ are homotopic. Alternatively, $j:X\to X^+$ has the characterizing property that for every map $g:X\to Z$, where Z is simply connected, there is a unique homotopy class of maps $g^+:X^+\to Z$ such that $g\simeq jg^+$. Furthermore, an X^+ may be obtained from X by attaching only 2-cells and 3-cells.

If now $f:X\to Y$ has the property that $G=f_*\pi_1 X$ is a normal subgroup of $\pi_1 Y$, then G is also perfect, and define Y_G^+ so that the following diagram is a homotopy cofiber pushout.

$$
\begin{array}{ccc}
X & \xrightarrow{\ j\ } & X^+ \\
{\scriptstyle f}\downarrow & & \downarrow \\
Y & \xrightarrow{\ k\ } & Y_G^+
\end{array}
$$

Take, e.g., Y_G^+ homotopy equivalent to the double mapping cylinder of $Y\leftarrow X\to X^+$. Then $\pi_1 Y_G^+=\pi_1 Y/G$ and $k_*:H_*(Y; E)\to H_*(Y_G^+; E)$ is an isomorphism for every $Z[\pi_1 Y/G]$-module E. The notation Y_G^+ for the above pushout suggests what is actually a fact—namely that Y_G^+ and k are homotopy unique, depending only upon the perfect normal subgroup $G\subset\pi_1 Y$ and not upon the choice of X and f. Moreover, given the perfect normal subgroup G of $\pi_1 Y$, appropriate $f:X\to Y$ exist to define Y_G^+; in fact, Y_G^+ may be obtained from Y by attaching only 2-cells and 3-cells. Also, $k:Y\to Y_G^+$ has a characterizing universal mapping property similar to the one described above for $j:X\to X^+$.

If $\phi:G\to A$ is a homomorphism of groups with G perfect and ϕG normal in A (of course, ϕG is also perfect), then there is the space $(BA)_{\phi G}^+$, where

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BA denotes a classifying space for the group A. If the homomorphism ϕ is understood, then we may also write $(BA)_G^+$ for $(BA)_{\phi G}^+$. The homotopy of $(BA)_{\phi G}^+$ has proved interesting for certain $\phi: G \to A$; for example,

$$\pi_*(K_0R \times (BGL(R))_{E(R)}^+) \cong K_*R, \quad \text{Quillen's higher algebraic } k\text{-theory},$$

$$\pi_*(Z \times (B\Sigma_\infty)_{A_\infty}^+) \cong \pi_*^S, \quad \text{stable homotopy of spheres (see [2])}.$$

DEFINITION.   If G is any perfect group, then define the *Schur space* of G to be

$$\text{Sch } G = (B \text{ Aut } G)_G^+,$$

where $G \to \text{Aut } G$ is the canonical homomorphism $g \mapsto$ inner automorphism induced by $g \in G$. The *Schur homotopy groups* of G are defined to be the groups

$$\text{Sch}_* G = \pi_* \text{ Sch } G.$$

The terminology just introduced is to some extent justified by the following, which follows directly from Quillen [1].

PROPOSITION.   *If G is a perfect group, then*

$$\text{Sch}_1 G \cong \text{Out } G = \text{Aut } G/(\textit{inner automorphisms}),$$
$$\text{Sch}_2 G \cong H_2(G/center; Z) = M(G/center)$$
$$= \textit{Schur multiplier of } G/center,$$
$$\text{Sch}_3 G \cong H_3(\tilde{G}; Z),$$

where $\tilde{G}$ denotes the largest perfect group such that there is a central extension $0 \to M \to \tilde{G} \to G \to 1$. In general,

$$\text{Sch}_n G \cong \text{Sch}_n \tilde{G} \cong \pi_n(BG)^+ \quad \textit{for } n \geqq 3.$$

(Note that $M = M(G) \cong H_2(G; Z) \cong \pi_2(BG)^+$ is the Schur multiplier of G.)

Understanding or computing higher Schur homotopy groups can be quite a problem. The only concrete examples are typified by $\text{Sch } A_\infty = (B\Sigma_\infty)_{A_\infty}^+$ with homotopy groups $\text{Sch}_* A_\infty \cong \pi_*^S$. Particularly intractible seem to be the Schur homotopy groups of finite perfect groups for which next to nothing is known. One problem in this situation is that there is no hope of utilizing infinite loop space (or generalized cohomology theoretic) techniques. In a paper to be published elsewhere, we intend to discuss some more or less systematic computational techniques which yield partial information on the Schur homotopy of finite groups.

For the remainder of this paper, we discuss in detail the Schur homotopy of the simplest possible perfect group A_5. We are fortunate that some

rather *ad hoc* methods may be used to obtain a reasonably good description of $\text{Sch}_* A_5$. From the Proposition above, we have immediately that $\text{Sch}_1 A_5 \cong Z_2$, $\text{Sch}_2 A_5 \cong Z_2$, and $\text{Sch}_3 A_5 \cong H_3(\Delta; Z) \cong Z_{120}$ where $\tilde{A}_5 \cong \Delta$ is the *binary icosahedral group* of order 120, presented by
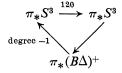
$$\Delta = gp\{x, y : x^5 = y^3 = (xy)^2\}$$

(see, for example, Coxeter-Moser [3, §6.5]). The center of Δ is of order 2 generated by $z = (xy)^2$, and setting $z = 1$ gives a classical presentation for A_5. There is a faithful, irreducible representation $\Delta \subsetneq SU(2) \cong S^3$ whose homogeneous space $\Sigma = S^3/\Delta$ is the Poincaré dodecahedral space. Now Σ has the same homology as the 3-sphere S^3. In fact, identifying the complement of a nice open 3-cell in Σ to a point gives a degree 1 map $j : \Sigma \to S^3$ such that $j_* : H_*(\Sigma; C) \to H_*(S^3; C)$ is an isomorphism for all C; in other words, $S^3 \simeq \Sigma^+$.

The representation $\Delta \subsetneq S^3$ induces a map on classifying spaces $p : B\Delta \to BS^3$, which may be taken to be a fibration with fiber Σ; equivalently, the total space E of the principal S^3 bundle over $B\Delta$ induced by p is canonically homotopy equivalent to Σ. But since BS^3 is simply connected, p factors homotopy uniquely through $(B\Delta)^+$ by a map $p^+ : (B\Delta)^+ \to BS^3$ with $p \simeq jp^+$. Hence, the principal S^3 bundle over $B\Delta$ mentioned above is induced by $j : B\Delta \to (B\Delta)^+$ from the principal S^3 bundle over $(B\Delta)^+$ induced by p^+. Since $j_* : H_*(B\Delta; C) \to H_*((B\Delta)^+; C)$ is an isomorphism for all C, then, by the comparison theorem for the Serre spectral sequence, and since S^3 and $(B\Delta)^+$ are simply connected, the total space of the bundle over $(B\Delta)^+$ is just $E^+ \simeq \Sigma^+ \simeq S^3$. The inclusion of fiber $S^3 \subsetneq E \simeq \Sigma$ has degree ± 120, and thus so does the inclusion of fiber $S^3 \subsetneq E^+ \simeq S^3$. In other words, we have a homotopy fibration

$$S^3 \xrightarrow{\ 120\ } S^3 \longrightarrow (B\Delta)^+.$$

Taking homotopy groups yields an exact couple of "Bockstein" type,

$$
\begin{array}{ccc}
\pi_* S^3 & \xrightarrow{\ 120\ } & \pi_* S^3 \\
& \text{degree } -1 \nwarrow \quad \swarrow & \\
& \pi_* (B\Delta)^+ &
\end{array}
$$

with $E_*^1 = \pi_*(B\Delta)^+$. Also, $\pi_n(B\Delta)^+ \cong \text{Sch}_n A_5$ for $n \geq 3$. Thus $\text{Sch}_4 A_5 \cong Z_2$, $\text{Sch}_5 A_5$ is of order 4, and $\text{Sch}_6 A_5$ is an extension of Z_{12} by Z_2. In general, $\text{Sch}_* A_5$ has no p-torsion for primes $p > 5$.

Deep results of James (for $p=2$) and Toda (for p odd) show that $\pi_* S^3$ has no elements of order p^3 for any prime p (see [4], [5]). Although there are plenty of elements of order 4, there is no *known* element of $\pi_* S^3$ of

order p^2 for p odd. It follows for $p=2$ and perhaps also for $p=3, 5$ that there are short exact sequences for $n>4$,

$$0 \to \pi_n S^3 \otimes Q_p \to \mathrm{Sch}_n A_5 \otimes Q_p \to \pi_{n-1} S^3 \otimes Q_p \to 0,$$

where Q_p denotes the localisation of Z at p.

Also, $E_n^2 \otimes Q_p$ vanishes for $n>3$, $p=2$ and, we conjecture, for $p=3, 5$ as well. Note that the 2-primary results are true by "overkill". The fact that we have multiplied by one more power of 2 than is necessary to kill the 2-primary part of $\pi_* S^3$ is enough to enable us to compute some (but unfortunately not all) of the extensions in the above sequence for $p=2$. In particular, we have $8 \mathrm{Sch}_* A_5 \otimes Q_2 = 0$, and if $2\pi_j S^3 \otimes Q_2 = 0$ for $j=n-1$ or n, then the sequence above splits $(n>4, p=2)$. Thus, we have

$$\mathrm{Sch}_5 A_5 \cong Z_2 \oplus Z_2, \quad \mathrm{Sch}_6 A_5 \cong Z_{12} \oplus Z_2, \quad \mathrm{Sch}_7 A_5 \cong Z_2 \oplus Z_{12}, \quad \text{etc.}$$

We have not yet found a nontrivial extension and suspect that every extension splits in the above sequence.

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