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REPRESENTATION OF PARTIALLY ORDERED LINEAR ALGEBRAS

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In [2] and [3] a condition on partially ordered linear algebras (*pola*'s) is defined, and it is shown that Dedekind σ-complete polas satisfying this condition have many of the properties of function spaces. Using a theorem of H. Nakano we can show, even without the hypothesis that the pola is Dedekind σ-complete, that any such pola is isomorphic to a pola of continuous, almost-finite, extended-real-valued functions. If A is a pola with multiplicative identity 1 the condition mentioned is:

P_1. If $x \in A$ and $x \geqq 1$, then x has an inverse and $x^{-1} \geqq 0$.

THEOREM. *In order for an Archimedean pola A with identity* 1 *to be isomorphic to a pola of continuous, almost-finite, extended-real-valued functions on a compact Hausdorff space X, it is sufficient that* P_1 *hold for A. The condition is necessary also if $A_1 = \{y \in A$: there exists $\alpha \in R^+$ with $-\alpha 1 \leqq y \leqq \alpha 1\}$ is complete in the order unit norm derived from* 1 *and if the image of A_1 separates points in X.*

PROOF. The standard completion procedure for Archimedean ordered linear spaces shows that A is isomorphic with an order dense subspace $\hat{A}$ of a Dedekind complete linear lattice D. In [**4**, p. 150] it is shown that the multiplication on $\hat{A}$ can be extended to D in such a way that D is a pola if the following continuity condition is satisfied: For every subset B of A, $\inf B = 0$ implies $\inf(aB) = \inf(Ba) = 0$ for all positive elements a in A. Given P_1, multiplication by $(a+1)^{-1}$ shows this condition is satisfied. Thus D is a linear lattice pola and the order density of $\hat{A}$ shows (since 1 is easily seen to be a weak order unit for A) that the image of 1 is a weak order unit for D. Now D (and hence A) has a representation of the type desired by [**1**, Corollary, p. 625].

To prove the second statement we note that the assumptions, together with the Stone–Weierstrass theorem, give the result that if $A \rightarrow \hat{A}$ is the isomorphism then $\hat{A}_1 = C(X)$. Then, given any x in A such that $x \geqq 1$,

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we can define an f in $C(X)$ by $f(t)=1/\hat{x}(t)$ for all t in X (with $1/\infty$ set equal to 0). Then there exists z in A_1 such that $\hat{z}=f$ and it is clear that $z=x^{-1}$ and $z\geqq 0$.

Note that it is not enough to know that A separates points of X to conclude that A_1 does. This shows the need for the separation assumption. Also, it is easy to see that if A is Dedekind σ-complete, then A_1 is complete in the order unit norm, so this case is included.

An immediate consequence of this theorem is the useful result that if a pola is Archimedean, has an identity, and satisfies P_1, then it is necessarily commutative.

References

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