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AN EMBEDDING-OBSTRUCTION FOR PROJECTIVE VARIETIES[1]

BY AUDUN HOLME

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A classical problem in differential topology is the following: Let X be a compact n-dimensional differentiable manifold (without boundary). Then compute the least integer $m=m(X)$ such that X may be embedded into $\boldsymbol{R}^m$. Usually this question is attacked as follows (see Atiyah [**1**]): (a) An upper bound for m is obtained by exhibiting explicit embeddings, and (b) a lower bound is obtained by certain homotopy invariants.

The forthcoming paper [**2**] deals with an algebro-geometric counterpart to the problem mentioned above: Let X be a nonsingular, projective k-variety embedded in some projective space $\boldsymbol{P}_k^N$ by the embedding i. For simplicity we assume the field k to be algebraically closed, but the results of [**2**] still hold under the weaker assumption that k is infinite. The main result is that the least integer $m=m(X, i)$, such that X can be embedded into $\boldsymbol{P}_k^m$ via a projection from $\boldsymbol{P}_k^N$, is *effectively computed in terms of the degrees of the Chern-classes of X.*

More precisely, let $X \subsetneq \boldsymbol{P}_k^N$ be an n-dimensional nonsingular projective variety, embedded in $\boldsymbol{P}_k^N$. Let $c_i=c_i(X)=c_i(\Omega^1_{X/k}) \in A(X)$ be the Chern-classes of X, where $A(X)$ denotes the Chow-ring of X. Consider the formal inverse of the alternating Chern-polynomial:

$$\left[\sum_{i=0}^{n}(-1)^i c_i T^i\right]^{-1} = \sum_{i=0}^{\infty} f_i T^i.$$

Here $f_i=0$ for $i>n$. Let $d_i=\deg(f_i)$ with respect to the embedding $i: X \subsetneq \boldsymbol{P}_k^N$. In particular $d_0=\deg(i(X))=d$. Define

$$B_X(T) = \left(\sum_{i=0}^{n} d_i T^i\right)\left(\sum_{i=0}^{2n+1}\binom{2n+2}{i}T^i\right) = B_0 + B_1 T + \cdots,$$

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of which we only need $B_0, B_1, \cdots, B_n$. In fact, we put

$$\beta_j = \sum_{i=0}^{j-n}(-1)^i \binom{n-i}{j-i-n}\left(B_i - d^2\binom{n+1}{i}\right), \qquad n \leqq j \leqq 2n,^2$$

$$\beta_j = 1 \quad \text{for } j < n, \qquad \beta_j = 0 \quad \text{for } j > 2n.$$

DEFINITION. *For all integers m the sequence* $(\beta_m, \beta_{m+1}, \cdots)$ *is called the mth embedding obstruction of the embedded variety* (X, i).

In [2] the following result is proved:

THEOREM. *If* $m<N$, *then X can be embedded into* $\boldsymbol{P}_k^m$ *via a projection from* $\boldsymbol{P}_k^N$ *if and only if the mth obstruction vanishes, i.e.*

$$(\beta_m, \beta_{m+1}, \cdots) = (0, 0, \cdots). \tag{1}$$

This implies at once the well-known and classical (see E. Lluis [5]):

COROLLARY. $m(X, i) \leqq 2n+1$.

For $n=1$ and $m=2$ we obtain the well-known *genus-formula*

$$g(X) = \tfrac{1}{2}(d-1)(d-2) \tag{2}$$

which is necessary and sufficient for when the nonsingular curve X can be projected isomorphically onto a plane curve. For $n=2$, $m=3$, we get that a nonsingular surface X in $\boldsymbol{P}_k^N$ can be embedded into $\boldsymbol{P}_k^3$ via a projection if and only if

$$\begin{aligned} \deg(K_X) &= (d-4)d, \\ (K_X^2) &= (d-4)^2 d, \\ p_a(X) &= \tfrac{1}{6}(d-1)(d-2)(d-3). \end{aligned} \tag{3}$$

Again $d=\deg(X)$, K_X is the canonical divisor and $p_a(X)$ the arithmetic genus of X. The necessity of (3) was noted by Iversen [4].

It should be easy to compute formulas similar to (2) and (3) in any dimension n by means of (1), and thus obtain a characterization (in terms of classical invariants like K_X, $p_a(X)$) of those nonsingular varieties X in $\boldsymbol{P}_k^N$ which can be projected isomorphically onto a hypersurface in $\boldsymbol{P}_k^{n+1}$. Of course (1) with $m=n+1$ gives such a characterization, in terms of the degrees of certain monomials in the Chern-classes of X.

Another application of the theorem is to Abelian varieties. In fact, the question of embeddings for Abelian varieties is resolved as follows: Let $X \subsetneq \boldsymbol{P}_k^N$ be an n-dimensional Abelian variety. Then:

(i) *X can always be embedded into* $\boldsymbol{P}_k^{2n+1}$ *via a projection from* $\boldsymbol{P}_k^N$;

(ii) *X can be embedded into* $\boldsymbol{P}_k^{2n}$ *via a projection from* $\boldsymbol{P}_k^N \Leftrightarrow \deg(X) = \frac{1}{2}\binom{2n+1}{n}$;

(iii) *X cannot be embedded into* $\boldsymbol{P}_k^{2n-1}$.

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[2] ADDED IN PROOF. Using standard combinatorial identities, one easily checks that $(-1)^{j-n}\beta_j = \left(\sum_{i=0}^{j-n} \binom{j+1}{j-n-1} d_i\right) - d_0^2$.

For $n=1$, (ii) gives $\deg(X)=3$ which is no surprise, and for $n=2$ we get $\deg(X)=10$. The necessity of this condition for the embedding of a 2-dimensional Abelian variety into $\boldsymbol{P}_k^4$ was noted by Horrocks and Mumford in [**3**, Theorems 5.1 and 5.2].

It should be noted that [**2**] deals only with *embedded projective varieties*. For a given projective variety X, one may ask for the least integer $e=e(X)$ such that X may be embedded into $\boldsymbol{P}_k^e$. If X is given as a subvariety of some $\boldsymbol{P}_k^N$, one may very well have $m(X)>e(X)$. Nevertheless, calculation of $m(X)$ can be used to obtain upper and lower bounds for $e(X)$, see for example the computation for Abelian varieties referred to above. In order to compute $e(X)$, one must find the projective embeddings i of X for which $m(X, i)$ is minimal, i.e., for which the embedding obstruction is as nice as possible. We hope to return to this question later.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BERGEN, BERGEN, NORWAY