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2^X AND $C(X)$ ARE HOMEOMORPHIC TO THE HILBERT CUBE[1]

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1. **Introduction.** Let 2^X be the hyperspace of nonempty closed subsets of a metric continuum X, and let $C(X)$ be the space of nonempty subcontinua of X, both with the Hausdorff metric. This paper is a description of the general techniques used in obtaining the following results.

THEOREM 1. $2^X \approx Q$, *the Hilbert cube, if and only if* X *is a nondegenerate Peano space* (*locally connected metric continuum*).

THEOREM 2. $C(X)\times Q\approx Q$ *if and only if* X *is a Peano space, and* $C(X)\approx Q$ *if and only if* X *is a nondegenerate Peano space containing no free arcs.*

Theorem 1 answers a question posed by Wojdyslawski [**8**], who later showed that 2^X is an AR for every Peano space X [**9**]. The converse is easily seen to be true; in fact, if 2^X is locally connected, then so is X. The proof of Theorem 1 is based on the recent result of Schori and West [**5**] that $2^\Gamma \approx Q$ for every nondegenerate connected graph Γ.

Wojdyslawski also showed that $C(X)$ is an AR if (and only if) X is a Peano space. An important special case of Theorem 2 is already known: if Γ is a connected graph, then $C(\Gamma)$ is a contractible polyhedron [**4**], and therefore $C(\Gamma)\times Q\approx Q$ by a theorem of West [**6**]. Since $C(I)\approx I^2$, the condition that X contains no free arcs is clearly necessary for $C(X)\approx Q$. The proof of sufficiency uses a recent result of West [**7**] that $C(D)\approx Q$ for every dendron D with a dense set of branch points.

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Certain relative versions of these theorems are also obtained. For $A \in 2^X$, let $2^X_A=\{B \in 2^X : A \subset B\}$, and for $A \in C(X)$, let $C_A(X)=\{B \in C(X) : A \subset B\}$.

THEOREM 3. *$2^X_A \approx Q$ if X is a Peano space and $A \neq X$. $C_A(X) \times Q \approx Q$ if X is a Peano space, and $C_A(X) \approx Q$ if X is a Peano space, $A \neq X$, and $X \backslash A$ contains no free arcs.*

2. **Outline of the proof for 2^X.** A map $f: Q_1 \rightarrow Q_2$ between copies of the Hilbert cube is a *near-homeomorphism* if it is the uniform limit of (onto) homeomorphisms. It is easily seen that such a map must be a monotone surjection.

APPROXIMATION LEMMA. *Let Y be a metric continuum, with $Q_1 \xleftarrow{f_1} Q_2 \xleftarrow{f_2} \cdots$ an inverse sequence of maps and copies of the Hilbert cube in Y such that*

(i) *$Q_i \rightarrow Y$ (in 2^Y),*

(ii) *$d(f_i, \mathrm{id}) < 2^{-i}$ for each i,*

(iii) *$\{f_i \circ \cdots \circ f_j : j \geqq i\}$ is an equi-uniformly continuous family for each i, and*

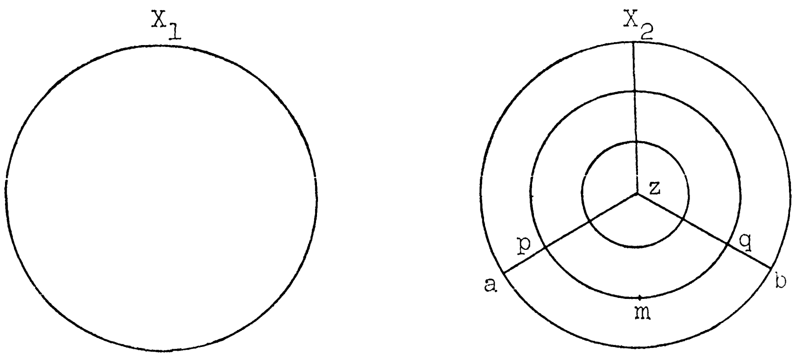
(iv) *each f_i is a near-homeomorphism. Then $Y \approx Q$.*

PROOF. Suppose first that each f_i is a homeomorphism. From (ii) we have for each $y \in Q_1$ that $\{(f_1 \circ \cdots \circ f_i)^{-1}(y) : i \geqq 1\}$ is a Cauchy sequence in Y. This together with (i) implies that $h: Q_1 \rightarrow Y$ defined by $h(y)=\lim_{i \rightarrow \infty}(f_1 \circ \cdots \circ f_i)^{-1}(y)$ is a continuous surjection. Then (iii) applied for $i=1$ shows that h is 1-1, and is therefore a homeomorphism.

In general, we inductively replace the near-homeomorphisms $\{f_i\}$ with homeomorphisms $\{g_i\}$ while retaining conditions (ii) and (iii) with respect to the $\{g_i\}$. (In doing this it is necessary to apply (iii) for each $i \geqq 1$.) This procedure is essentially that used by Brown [3] to show that if each f_i is a near-homeomorphism, then inv $\lim(Q_i, f_i) \approx Q$.

We apply the approximation lemma to the hyperspace 2^X of a Peano space X by constructing an inverse sequence $2^{\Gamma_1} \xleftarrow{f_1} 2^{\Gamma_2} \xleftarrow{f_2} \cdots$, where $\{\Gamma_i\}$ is a sequence of connected graphs in X converging to X (thus each $2^{\Gamma_i} \approx Q$ and $2^{\Gamma_i} \rightarrow 2^X$), and the maps $\{f_i\}$ are near-homeomorphisms satisfying (ii) and (iii) of the lemma.

While in general we use partitions of X to construct the connected graphs $\{\Gamma_i\}$ (see §4), in the special case where X is polyhedral they are more readily obtained as the 1-skeletons of subdivisions $\{X_i\}$ of X, where X_{i+1} is a subdivision of X_i for each i and mesh $X_i \rightarrow 0$. Each map $f_i: 2^{\Gamma_{i+1}} \rightarrow 2^{\Gamma_i}$ is induced by a map $\varphi_i: \Gamma_{i+1} \rightarrow C(\Gamma_i)$ (i.e., $f_i(A)=\bigcup\{\varphi_i(x): x \in A\}$). Conditions (ii) and (iii) are satisfied by inductively requiring at each stage that mesh X_i be sufficiently small.



Γ_1 = boundary Γ_2 = 1-skeleton

FIGURE 1

We illustrate this construction for a 2-cell X. The subdivision X_2 may be made arbitrarily fine by using as many concentric circles and as many radii as necessary. Each succeeding subdivision X_{i+1} will be obtained by subdividing in a (topologically) similar fashion each 2-cell of X_i.

The map $\varphi_1:\Gamma_2\to C(\Gamma_1)$ is defined as follows:

(i) $\varphi_1(x)=\{x\}$, $x\in\Gamma_1$,

(ii) $\varphi_1(z)=\Gamma_1$,

(iii) φ_1 is a linear expansion along each radius $\overline{az}$, $\overline{bz}$, etc.,

(iv) $\varphi_1(m)=\varphi_1(p)\cup\overline{ab}\cup\varphi_1(q)$, etc.,

(v) φ_1 is a linear expansion along each circular arc $\overline{pm}$, $\overline{qm}$, etc.

Succeeding maps $\varphi_i:\Gamma_{i+1}\to C(\Gamma_i)$ are defined similarly.

The above map illustrates the more general notion of a piecewise-linear map $\varphi:M\to C(N)$, for connected graphs M and N. In a forthcoming paper we formally define this notion and prove, as a major part of the paper, that the map $f:2^M\to 2^N$ induced by the piecewise-linear map φ is a near-homeomorphism if f is a monotone surjection.

REMARK ON φ_1. The most obvious candidate for this map is $\varphi_1=h/\Gamma_2$, where $h:X\to C(\Gamma_1)$ is the well-known homeomorphism of a 2-cell onto the hyperspace of subcontinua of its boundary, defined by

$$h(re^{i\theta}) = [e^{i(\theta-(1-r)\pi)}, e^{i(\theta+(1-r)\pi)}].$$

But the corresponding induced map f_1 fails to be monotone; for the subdivision X_2 illustrated above, there exist arcs A on Γ_1 for which $f_1^{-1}(A)\subset 2^{\Gamma_2}$ has three components.

Suppose now that the subdivisions $X_1, \cdots, X_i$ and the corresponding maps $\varphi_1, \cdots, \varphi_{i-1}$ have been selected, with mesh $X_j<2^{-j}$ for each j. For $1\leqq m<n$, define $f_m^n:2^{\Gamma_n}\to 2^{\Gamma_m}$ by $f_m^n=f_m\circ\cdots\circ f_{n-1}$. Now, choose

$0<\delta_i<1/i$ such that for $A, B \in 2^{\Gamma_i}$ with $\rho(A, B)<\delta_i$, we have $\rho(f_j^i(A), f_j^i(B))<1/i$ for $1 \leqq j \leqq i-1$. We take a subdivision X_{i+1} of X_i with mesh $X_{i+1}<2^{-(i+1)}$, and such that for points x, y on the boundary of any 2-cell of X_{i+1}, we have $\rho(\varphi_i(x), \varphi_i(y))<\delta_i/2$. (Clearly, this type of condition is achievable for the subdivision X_2 and the map φ_1, and as previously noted each map $\varphi_i:\Gamma_{i+1}\to C(\Gamma_i)$ is defined in a similar fashion.)

We now prove that this construction of the inverse sequence $2^{\Gamma_1} \xleftarrow{f_1} 2^{\Gamma_2} \xleftarrow{f_2} \cdots$ satisfies (ii) and (iii) of the lemma. For $x \in \Gamma_{i+1}$ we have $\varphi_i(x)\subset\dot{\sigma}$, for σ a 2-cell in X_i containing x, and since mesh $X_i<2^{-i}$ it follows that $d(f_i, \mathrm{id})<2^{-i}$. To verify (iii), let $\varepsilon>0$ and $k \geqq 1$ be given. Choose $j \geqq k$ such that $1/j<\varepsilon$. Choose $\mu>0$ such that for $x, y \in X$ with $d(x, y)<\mu$, there exist intersecting 2-cells σ_x and σ_y of X_{j+1} containing x and y, respectively. Now consider points $x, y \in \Gamma_i$, $i \geqq j+1$, with $d(x, y)<\mu$. With σ_x and σ_y as above, we have $f_{j+1}^i(\{x\})\subset\dot{\sigma}_x$ and $f_{j+1}^i(\{y\})\subset \dot{\sigma}_y$, and it follows from the construction of X_{j+1} and φ_j that $\rho(f_j^i(\{x\}), f_j^i(\{y\}))<\delta_j$. Thus for $A, B \in 2^{\Gamma_i}$, $i \geqq j+1$, with $\rho(A, B)<\mu$, we have $\rho(f_j^i(A), f_j^i(B))<\delta_j$, and therefore $\rho(f_k^i(A), f_k^i(B))<1/j<\varepsilon$.

3. **Modifications for** $C(X)$. The result $C(X)\times Q\approx Q$ is obtained by considering the inverse sequence

$$C(\Gamma_1) \times Q \xleftarrow{f_1\times \mathrm{id}} C(\Gamma_2) \times Q \xleftarrow{f_2\times \mathrm{id}} \cdots,$$

where the graphs $\{\Gamma_i\}$ are those constructed above and the maps $\{f_i\}$ are induced by the maps $\{\varphi_i\}$. Our techniques on piecewise-linear maps show that each map $f_i\times \mathrm{id}: C(\Gamma_{i+1})\times Q\to C(\Gamma_i)\times Q\approx Q$ is a near-homeomorphism, and these maps clearly satisfy (ii) and (iii) of the approximation lemma. Since $C(\Gamma_i)\times Q\to C(X)\times Q$, we have $C(X)\times Q\approx Q$.

To obtain the stronger result $C(X)\approx Q$, for X a nondegenerate polyhedron containing no free arcs, we proceed basically as before in the construction of the subdivisions $\{X_i\}$, but add at the ith stage of the construction finite collections of stickers to Γ_i and to each of its predecessors $\Gamma_{i-1}, \cdots, \Gamma_1$. These stickers are obtained from Γ_{i+1}, and do not alter the homology of the graphs $\Gamma_i, \cdots, \Gamma_1$. In this manner we eventually add countably many stickers to each Γ_i, obtaining (upon forming the closures) a sequence $\{\Gamma_i^*\}$ of connected local dendra with dense sets of branch points (there are no free arcs). West's techniques [7] are easily applied to this situation, yielding $C(\Gamma_i^*)\approx Q$. Furthermore, each $\Gamma_i^*\subset\Gamma_{i+1}^*$ and there exist monotone retractions $\{r_i:\Gamma_i^*\to\Gamma_i\}$.

There exist piecewise-linear maps $\{\gamma_i:\Gamma_{i+1}\to C(\Gamma_i^*\cap\Gamma_{i+1})\}$, similar to the maps $\{\varphi_i\}$, with $\gamma_i(x)=\{x\}$ for $x \in \Gamma_i^* \cap \Gamma_{i+1}$. A sequence of maps $\{\gamma_i^*:\Gamma_{i+1}^*\to C(\Gamma_i^*)\}$ is then obtained by extending the maps $\{\gamma_i\}$. Specifically, we set $\gamma_i^*(x)=\{x\}$ for $x \in \Gamma_i^*$, and $\gamma_i^*(x)=\gamma_i\circ r_{i+1}(x)$ otherwise.

These maps $\{\gamma_i^*\}$ induce near-homeomorphisms $\{g_i^*: C(\Gamma_{i+1}^*) \rightarrow C(\Gamma_i^*)\}$, and the inverse sequence $C(\Gamma_1^*) \xleftarrow{g_1^*} C(\Gamma_2^*) \xleftarrow{g_2^*}$ satisfies (ii) and (iii) of the approximation lemma and thus $C(X) \approx Q$.

4. In the general case we must construct connected graphs in a Peano space X. We do this by partitioning X, i.e., breaking up the space into a finite number of small, connected, and locally connected subsets intersecting only along their boundaries, in much the same way that a 2-cell is subdivided (see Bing [1], [2]). The boundaries of these partition elements will be accessible from the interiors, and this together with the arcwise connectivity of the Peano spaces enables us to construct trees in each element such that the union of all these trees is a connected graph Γ, which can be viewed as a 1-dimensional nerve of the partition of X. In this way we construct a sequence $\{P_i\}$ of partitions of X, with each P_{i+1} a refinement of P_i and mesh $P_i \rightarrow 0$, and a corresponding sequence $\{\Gamma_i\}$ of nerves of the partitions. The hyperspace maps $\{f_i\}$ are induced by piecewise-linear maps $\{\varphi_i\}$ which are constructed by a procedure similar to, but technically more difficult than, that employed in the special case where X is a polyhedron.

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