BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 5, September 1974

# $P_n$-SPACES AND $n$-FOLD LOOP SPACES

BY PETER V. Z. COBB

Communicated by S. Eilenberg, March 5, 1974

The purpose of this paper is to present a characterization of $n$-fold loop spaces for $1 \leqq n < \infty$. The approach is in the same spirit as G. Segal's investigation of infinite loop spaces via "special $\Gamma$-spaces" [**4**]. Category theoretic terminology not explained here may be found in [**1**].

**I. The $P$-construction on small pointed categories.** Let $P_1$ be the category with objects the finite ordered sets, $\boldsymbol{n}=\{0, \cdots, n\}$, and with morphism sets $P_1(\boldsymbol{n}, \boldsymbol{m})=\{f:\boldsymbol{n}\to\boldsymbol{m} \mid f(0)=0; f(i)\leqq f(j) \text{ if } i<j \text{ and } f(j)\neq 0\}$. Let $\#:P_1\times P_1\to P_1$ be the bifunctor such that $\boldsymbol{n}\#\boldsymbol{m}=\{0, \cdots, n+m\}$ and such that if $f_i \in P_1(\boldsymbol{n}_i, \boldsymbol{m}_i)$ for $i=1, 2$,

$$\begin{aligned} f_1\#f_2(j) &= f_1(j), & & 0 \leqq j \leqq n_1; \\ &= f_2(j-n_1)+m_1, & & n_1 < j \leqq n_1+n_2 \text{ and } f_2(j-n_1)\neq 0; \\ &= 0, & & n_1 < j \leqq n_1+n_2 \text{ and } f_2(j-n_1)=0. \end{aligned}$$

Then $\#$ is strictly associative and $\boldsymbol{0}$ is a two-sided unit for $\#$ and a unique null-object for $P_1$.

Let $C$ be a small category with a unique null-object $e$. For each $a \in C$, we will denote by $N_a$ and $O_a$ the unique morphisms in $C(a, e)$ and $C(e, a)$ respectively. We now construct a strictly monoidal category $P(C)$, which one might describe as a "wreath-product" of $P_1$ with $C$.

The objects of $P(C)$ are the finite sequences, $\langle a_1, \cdots, a_n\rangle$, of nonnull objects of $C$ (including the empty sequence $\langle\ \rangle$). If $\alpha=\langle a_1, \cdots, a_n\rangle$ and $\beta=\langle b_1, \cdots, b_k\rangle$, we set

$$P(C)(\alpha, \beta) = \{(f; h_1, \cdots, h_n) \mid f\in P_1(\boldsymbol{n}, \boldsymbol{k}), h_i \in C(a_i, b_{f(i)})\}.$$

(By convention, $b_0=e$.) Composition of morphisms is defined according to the rule:

$$(f'; h'_1, \cdots, h'_k)(f; h_1, \cdots, h_n) = (f'f; h'_{f(1)}h_1, \cdots, h'_{f(n)}h_n).$$

---

*AMS (MOS) subject classifications* (1970). Primary 18D10, 55D35; Secondary 55J10.

We define a bifunctor $\#: P(C)\times P(C)\to P(C)$ by:

$$\langle a_1, \cdots, a_n\rangle \# \langle b_1, \cdots, b_k\rangle = \langle a_1, \cdots, a_n, b_1, \cdots, b_k\rangle,$$

and

$$(f; h_1, \cdots, h_n) \# (f'; h'_1, \cdots, h'_k) = (f \# f' : h_1, \cdots, h_n, h'_1, \cdots, h'_k).$$

$\#$ is strictly associative and $\langle\ \rangle$ is a two-sided unit for $\#$ and a unique null-object. There is a natural embedding of $C$ as a full subcategory of $P(C)$ via the functor $e\mapsto\langle\ \rangle$; $a\mapsto\langle a\rangle$, if $a\in C$, $a\neq e$; $h\mapsto(I_1; h)$ if $h$ is a morphism in $C$.

If we let $P_0$ denote the full subcategory of $P_1$ containing just $\mathbf{0}$ and $\mathbf{1}$, it is easy to check that $P_1\cong P(P_0)$.

II. **Homotopy-monoidal functors.** Let $\tau$ denote the category of pointed, compactly generated topological spaces of the homotopy type of a CW-complex and all continuous basepoint-preserving maps. Let $\prod:\tau\times\tau\to\tau$ denote the direct product bifunctor. If $F$ is any functor from $P(C)$ to $\tau$, there is a natural transformation

$$L^F: F\cdot\#\dot{\to}\prod\cdot(F\times F): P(C)\times P(C)\to\tau$$

where, for $\alpha, \beta\in P(C)$, $L^F_{(\alpha,\beta)}: F(\alpha\#\beta)\to F(\alpha)\times F(\beta)$ is the unique map whose projections onto $F(\alpha)$ and $F(\beta)$ are $F(I_\alpha\# N_\beta)$ and $F(N_\alpha\# I_\beta)$ respectively. Notice that for $\alpha, \beta, \gamma\in P(C)$,

$$(L^F_{(\alpha,\beta)}\times I_{F(\gamma)})\cdot L^F_{(\alpha\#\beta,\gamma)} = (I_{F(\alpha)}\times L^F_{(\beta,\gamma)})\cdot L^F_{(\alpha,\beta\#\gamma)},$$

and that therefore, $L^F$ extends naturally to products of more than two elements. In particular, if $a_1, \cdots, a_n$ are in $C$ and $\alpha=\langle a_1, \cdots, a_n\rangle$, we have a map: $L^F_\alpha: F(\alpha)\to\prod_{i=1}^n F(a_i)$.

The functor $F$ is said to be *homotopy-monoidal* if $L^F$ is a natural homotopy equivalence; or equivalently, if $L^F_\alpha$ is a homotopy-equivalence for all $\alpha$ in $P(C)$. The category of all such homotopy-monoidal functors from $P(C)$ to $\tau$ will be denoted by $(P(C), \tau)_h$.

Let $R^+$ denote the topological monoid of nonnegative integers under addition. We let $\mathscr{M}_{R^+}$ denote the category of topological monoids over $R^+$. To be precise, an object of $\mathscr{M}_{R^+}$ is a pair $(M, q_M)$, where $M$ is a monoid in $\tau$, and $q_M$ is a continuous homomorphism of monoids from $M$ to $R^+$. A morphism from $(M, q_M)$ to $(M', q_{M'})$ is a continuous homomorphism $g: M\to M'$ such that $q_{M'}g=q_M$. The direct product in $\mathscr{M}_{R^+}$ is the pull-back over $R^+$, and we will denote it by the symbol $\overline{\prod}$. If $(C, e)$ is as above, we will let $(C, \mathscr{M}_{R^+})_0$ denote the category of functors from $C$ to $\mathscr{M}_{R^+}$ such that $F(e)=(R^+, I_{R^+})$ and $F(N_a)=q_{F(a)}$ for all $a\in C$.

THEOREM 1. *There is a functor,* $F \mapsto \hat{F}: (C, \mathcal{M}_{R^+})_0 \to (P(C), \tau)_h$ *such that* $\hat{F}|_C = |F|$, *where* $|\ |: \mathcal{M}_{R^+} \to \tau$ *is the forgetful functor.*

PROOF. Let $F \in (C, \mathcal{M}_{R^+})_0$. For $\alpha = \langle a_1, \cdots, a_n \rangle$ in $P(C)$ we set $\hat{F}(\alpha) = \coprod_{j=1}^n F(a_j)$. For $(f; h_1, \cdots, h_n) \in P(C)(\alpha, \beta)$, define $\hat{F}((\tau; h_1, \cdots, h_n))$ to be the composition:

$$\coprod_{i=1}^{n} F(a_i) \xrightarrow{\coprod(Fh_i)} \coprod_{i=1}^{n} F(b_{f(i)}) \simeq \coprod_{j=1}^{k} \left( \coprod_{f(i)=j} F(b_{f(i)}) \right) \xrightarrow{\coprod \mu_j} \coprod_{j=1}^{k} F(b_j),$$

where $\mu_j = F(O_{b_j})$ if $f^{-1}(j) = \varnothing$; $= I_{F(b_j)}$ if $f^{-1}(j)$ is singleton; and is the multiplication in $F(b_j)$ otherwise. That $\hat{F}$ is a functor is a straightforward but tedious exercise which we omit. ^ is defined on natural transformations in the obvious way and again we omit the details. It remains to verify that $\hat{F}$ is homotopy-monoidal.

If $\alpha = \langle a_1, \cdots, a_n \rangle \in P(C)$, then $L_\alpha^F : F(\alpha) \to \prod_{i=1}^n \hat{F}(a_i)$ is in fact the canonical inclusion $\coprod_{i=1}^n F(a_i) \subseteq \prod_{i=1}^n F(a_i)$. Define a homotopy-inverse to $L_\alpha^F$ as follows: Let $\mu_i$ denote the multiplication in $F(a_i)$, and if $x = (x_1, \cdots, x_n) \in \prod_{i=1}^n F(a_i)$, let $m_x = \max\{q_{F(a_i)}(x_i) | 1 \leqq i \leqq n\}$. Define $G(x) = (y_1, \cdots, y_n)$, where

$$y_i = \mu_i(F(O_{a_i})(m_x - q_{F(a_i)}(x_i)), x_i).$$

Since $q_{F(a_i)}(y_i) = m_x$ for all $i$, $(y_1, \cdots, y_n) \in \coprod_{i=1}^n F(a_i)$. $G$ is clearly a left inverse for $L_\alpha^F$ and a right homotopy-inverse for $L_\alpha^F$. Hence $F$ is homotopy-monoidal, and the theorem is proved.

## III. Special $P_n$-spaces and iterated loop spaces.

Following the pattern for $P_0$ and $P_1$ above, we define $P_n = P(P_{n-1})$ for $n \geqq 2$, and if $m < n$, we identify $P_m$ with its image in $P_n$. Notice that if $m < n$, $P_m$ is a full subcategory of $P_n$, but the monoid structure of $P_m$ is not related to the monoid structure of $P_n$, and hence, if $F \in (P_n, \tau)_h$, $F|_{P_m}$ is not necessarily in $(P_m, \tau)_h$. If $F$ is a functor from $P_n$ to $\tau$, we say that $F$ is a *special* $P_n$*-space* if $F|_{P_m}$ is in $(P_m, \tau)_h$ for all $m$, $1 \leqq m \leqq n$, and if $F(\mathbf{0})$ is a point. We denote the category of special $P_n$-spaces by $(P_n, \tau)_s$. If $X \in \tau$, we say that $X$ admits a *special* $P_n$*-structure* if there is an $F$ in $(P_n, \tau)_s$ such that $F(\mathbf{1}) \simeq X$.

If $F \in (P_n, \tau)_s$, we say that $F$ is *well-pointed*, if for every $\alpha \in P_n$, $F(O_\alpha)$: $F(\mathbf{0}) \to F(\alpha)$ is a cofibration. $(P_n, \tau)_{sw}$ will denote the category of well-pointed, special $P_n$-spaces.

THEOREM 2. *For every* $n \geqq 0$, *there is a functor* $W: (P_n, \tau)_{sw} \to (P_{n+1}, \tau)_{sw}$ *such that for every* $F \in (P_n, \tau)_{sw}$, $WF(\mathbf{1}) \simeq |\Omega F(\mathbf{1})|$, *where* $\Omega$ *is the Moore loop space functor*: $\tau \to \mathcal{M}_{R^+}$.

PROOF. Let $F \in (P_n, \tau)_{sw}$. Then $\Omega F \in (P_n, \mathcal{M}_{R^+})_0$ and, by Theorem 1, we have $(\Omega F)^\wedge \in (P_{n+1}, \tau)_h$. If $a \in P_n$, then $F(O_a): F(\mathbf{0}) \to F(a)$ is a cofibration, hence $\Omega F(O_a): \Omega F(\mathbf{0}) = R^+ \to \Omega F(a)$ is a cofibration, and it follows easily that $F(O_\alpha): R^+ \to (\Omega F)^\wedge(\alpha)$ is a cofibration for all $\alpha \in P_{n+1}$. We now define $WF(\alpha) = (\Omega F)^\wedge(\alpha)/R^+$, where the quotient is as spaces *not* as monoids. If $h \in P_{n+1}(\alpha, \beta)$, then $O_\beta h = O_\alpha$, hence $WF$ extends naturally to morphisms. The quotient natural transformation from $(\Omega F)^\wedge$ to $WF$ is a natural homotopy equivalence since $R^+$ is a cofibered contractible subset of $F(\alpha)$ for all $\alpha \in P_{n+1}$. Therefore $WF|_{P_m} \in (P_m, \tau)_h$ if $(\Omega F)^\wedge|_{P_m}$ is. By Theorem 1, $(\Omega F)^\wedge \in (P_{n+1}, \tau)_h$ and $(\Omega F)^\wedge|_{P_m} = |\Omega F||_{P_m}$ if $m \leqq n$. But $|\Omega|$ preserves homotopy equivalences and products up to homotopy equivalence, and it follows easily that $|\Omega F||_{P_m} \in (P_m, \tau)_h$, for $1 \leqq m \leqq n$. The theorem now follows immediately.

COROLLARY 2.1. *If $X \in \tau$ and $X$ has a cofibered basepoint, then $\Omega^n(X)$ admits a $P_n$-structure.*

The proof is an easy induction using Theorem 2.

IV. **Delooping.** We utilize the delooping technique of Segal [**4**] to prove that every connected space which admits a special $P_n$-structure is of the homotopy type of an $n$-fold loop space.

Recall that a semisimplicial object in $\tau$ is a functor $A: \Delta^{\mathrm{op}} \to \tau$, where $\Delta$ is the category whose objects are the finite ordered sets, $[n] = \{0, \cdots, n\}$ for $n \geqq 0$, and whose morphisms are all weakly increasing set functions. For $1 \leqq i \leqq n$, let $\lambda_i^n: [1] \to [n]$ be the map which sends 0 and 1 to $i-1$ and $i$ respectively. If $A$ is a semisimplicial object in $\tau$, we say that $A$ is a *special* $\Delta$-*space* if $A([0])$ is a point and the map $A([n]) \to (A([1]))^n$ induced by the maps $A(\lambda_i^n)$, for $1 \leqq i \leqq n$, is a homotopy equivalence for all $n \geqq 1$. If $A$ is a special $\Delta$-space, we let $BA$ denote the Milnor realization of $A$ as a semisimplicial space [**2**], [**3**], [**4**]. If $A([\mathbf{1}])$ is connected, then $BA$ is also connected, and Segal has proved [**4**] that $A([\mathbf{1}]) \simeq \Omega BA$.

THEOREM 3. *There is a functor $\bar{B}: (P_n, \tau)_s \to (P_{n-1}, \tau)_s$, for each $n \geqq 1$, such that if $F \in (P_n, \tau)_s$ and $F(\mathbf{1})$ is connected, then $\bar{B}F(\mathbf{1})$ is connected and $F(\mathbf{1}) \simeq \Omega \bar{B}F(\mathbf{1})$.*

PROOF. Define a functor $E: \Delta^{\mathrm{op}} \to P_1$ as follows: $E([n]) = \mathbf{n}$, and if $f \in \Delta([n], [m])$, then $Ef \in P_1(\mathbf{m}, \mathbf{n})$ is defined by:

$$\begin{aligned} Ef(i) &= 0, \quad i > f(n) \text{ or } i \leqq f(0), \\ &= j, \quad f(j-1) < i \leqq f(j). \end{aligned}$$

If $\lambda_i^n \in \Delta([1], [n])$ is as above, $1 \leqq i \leqq n$, then $E\lambda_i^n(j) = \delta_{ij}, j \in \mathbf{n}$. It follows that if $F$ is a special $P_1$-space, then $FE$ is a special $\Delta$-space.

Now, let $F \in (P_n, \tau)_s$. We have a functor, $\tilde{F}: P_{n-1} \to (P_1, \tau)_s$, given by: $\tilde{F}(a)(\mathbf{m}) = F(\langle a, \cdots, a\rangle)$ ($m$ terms), $\tilde{F}(a)(f) = F((f; I_a, \cdots, I_a))$, and $\tilde{F}(h)(\mathbf{m}) = F((I_m; h, \cdots, h))$, where $a$ is an object in $P_{n-1}$, $\mathbf{m}$ is an object in $P_1$, $f$ is a morphism in $P_1$ and $h$ a morphism in $P_{n-1}$. Using the functoriality of the realization functor $B$, we can define a functor $\bar{B}F:P_{n-1}\to\tau$ by $\bar{B}F(a)=B(\tilde{F}(a)E)$. A tedious but straightforward argument, using the fact that $B$ preserves products (of semisimplicial spaces) and homotopy equivalences, tells us that $\bar{B}F$ is a special $P_{n-1}$-space. We omit the details of this and of the equally straightforward account of the functoriality of $\bar{B}$.

Since $F(\mathbf{1})=\tilde{F}(\mathbf{1})(\mathbf{1})$, it follows from the remarks just preceding the statement of this theorem that $\bar{B}F(\mathbf{1})$ is connected and $F(\mathbf{1})\simeq\Omega BF(\mathbf{1})$, if $F(\mathbf{1})$ is connected.

COROLLARY 3.1. *Suppose $X$ is in $\tau$, is connected, and admits a special $P_n$-structure. Then there exists a $Y$ in $\tau$ such that $X\simeq\Omega^n(Y)$.*

The proof is an easy induction using Theorem 3.

## BIBLIOGRAPHY

1. S. Mac Lane, *Categories for the working mathematician*, Springer-Verlag, New York, 1971.

2. J. Milnor, *The geometric realization of a semi-simplicial complex*, Ann. of Math. (2) **65** (1957), 357–362. MR **18**, 815.

3. G. Segal, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci. Publ. Math. No. 34 (1968), 105–112. MR **38** #718.

4. ———, *Homotopy-everything H-spaces* (preprint).

DEPARTMENT OF MATHEMATICS, FORDHAM UNIVERSITY, NEW YORK, NEW YORK 10458