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ON IDEALS OF COMPACT OPERATORS

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In [1] Brown, Pearcy and Salinas give an affirmative answer to the following question: Given a compact operator T on a separable Hilbert space H, is there an ideal $\Lambda(T)$ containing T and different from the ideal K of all compact operators? Their construction relies on some ideas of Von Neumann-Calkin [2] and is rather complicated.

The purpose of this brief note is to show that the existence of such a $\Lambda(T)$ follows from elementary properties of the s-numbers of T. Recall that the s-numbers, $(s_n(T))$, are the eigenvalues of $(TT^*)^{1/2}$ arranged in decreasing order and counting multiplicities. We list the three properties we need:

(1) If $S, T \in K$, then $s_{n+m}(S+T) \leqq s_n(S)+s_m(T)$.

(2) If $R, S, T \in L$, then $s_n(RST) \leqq \|R\| s_n(S) \|T\|$. Here L denotes the space of all bounded linear operators on H.

(3) If $T \in K$, then there are orthonormal sets (f_n) and (y_n) in H such that $T = \sum_{n=1}^{\infty} s_n(T) f_n \otimes y_n$.

We also use the following fact concerning real sequences:

(4) If (β_n) is a nonnegative sequence of real numbers increasing to ∞ with $1/\beta_n \in c_0 \backslash \bigcup_{p>0} l_p$, then there is a positive sequence $(\alpha_n) \in l_1$ such that $\sum_{n=1}^{\infty} \alpha_n \beta_n = +\infty$ and $(\beta_n \alpha_n)$ is decreasing. Also, there is a decreasing null sequence (γ_n) such that $\sum_{n=1}^{\infty} \gamma_n \alpha_n \beta_n = +\infty$.

The construction of $\Lambda(T)$. Let $\sigma_p = \{T \in K : \sum_{n=1}^{\infty} s_n(T)^p < +\infty\}$. It is well known and easy to prove that $K \backslash \bigcup_{p>0} \sigma_p \neq \varnothing$. (For a study of these important ideals see [3] and [4].) Thus we may suppose, for our purpose, that $T \in K \backslash \bigcup_{p>0} \sigma_p$. Then $\beta_n = 1/s_n(T)$ increases to ∞ and $1/\beta_n \in c_0 \backslash \bigcup_{p>0} l_p$. Let (α_n) be as in (4) and let

$$\Lambda(T) = \left\{ S \in L : \sum_{n=0}^{\infty} s_n(S) \alpha_n \beta_n < +\infty \right\}.$$

It follows from (4) and the definition of (β_n) that $T \in \Lambda(T)$.

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If the rank of S is finite, then $s_n(S)=0$ for large enough n and so $\Lambda(T)$ contains the finite rank operators (this fact also follows from deeper considerations).

If $R, S \in \Lambda(T)$ then by (1) and (4)

$$\alpha_{2n}\beta_{2n}s_{2n}(R+S) \leqq \alpha_n\beta_n s_n(R) + \alpha_n\beta_n s_n(S),$$

and similarly for $\alpha_{2n-1}\beta_{2n-1}s_{2n-1}(R+S)$. Thus $\Lambda(T)$ is a linear space. Also, it follows from (2) that $\Lambda(T)$ is closed under left and right composition by bounded linear operators.

Let $T_0=\sum \gamma_n f_n \otimes y_n$, where (γ_n) is as in (4) and (f_n) and (y_n) are determined by (3). Then $s_n(T_0)=\gamma_{n+1}$ and so by (4) $T_0 \in K\backslash\Lambda(T)$.

Using the recent results [5] and [6] it is easy to generalize the above construction to include many other classes of Banach spaces.

Recall that the nth approximation numbers, $\alpha_n(T)$, of T are given by

$$\alpha_n(T) = \inf\{\|T-A\| : \operatorname{rank} A \leqq n\}.$$

For Banach spaces E, F we say $T \in l_p(E, F)$ if $\sum_{n=1}^{\infty} \alpha_n(T)^p < +\infty$. These generalized ideals have been studied by Pietsch and others. The only fact we need here is the following [5]: *If* $K(E, F)=l_p(E, F)$ *for some* $p>0$, *then* $\min(\dim E, \dim F) < +\infty$. Here $K(E, F)$ denotes the compact operators from E to F. We also recall the following definition [6]: Two Banach spaces E and F form a *Bernstein pair* if for any positive, decreasing null sequence (b_n) there is a $T \in K(E, F)$ such that

$$0 < \inf_n \frac{\alpha_n(T)}{b_n} \leqq \sup_n \frac{\alpha_n(T)}{b_n} < +\infty. \tag{5}$$

Let $\mathscr{K}$ denote the ideal of all compact operators between arbitrary Banach spaces and let $T \in \mathscr{K}$.

THEOREM. *There exists a complete quasi-normed ideal* $\Lambda(T)$ *such that* $\Lambda(T)(E, F) \neq K(E, F)$ *whenever* $\langle E, F\rangle$ *forms a Bernstein pair.*

Indeed, by the result of Pietsch, we may assume that $(\alpha_n(T)) \in c_0\backslash\bigcup_{p>0} l_p$. Let $\beta_n=1/\alpha_n(T)$ and let α_n, γ_n be as in (4) above. Let

$$\Lambda(T) = \left\{S \in \mathscr{K} : \sum_{n=1}^{\infty} \alpha_n(S)\alpha_n\beta_n < +\infty\right\}.$$

Since the approximation numbers have properties (1)–(3), $\Lambda(T)$ is ideal. It is easy to show that under the quasi-norm

$$\rho(S) = \sum_{n=1}^{\infty} \alpha_n(S)\alpha_n\beta_n,$$

$\Lambda(T)$ is complete. If $\langle E, F\rangle$ is a Bernstein pair, then by (5) there is an $S \in K(E, F)$ such that $\inf_n(\alpha_n(S)/\gamma_n)>0$ and thus, by (4), $S \notin \Lambda(T)(E, F)$.

We remark that all classical Banach spaces form Bernstein pairs. In particular $\langle L_p(\mu), L_q(\gamma)\rangle$ is a Bernstein pair for all $1 \leqq p, q \leqq \infty$ and measures μ, γ [**6**].

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