BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 5, September 1974

FACTORIZATION AND INVARIANT SUBSPACES FOR NONCONTRACTIONS

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Communicated by Mary Ellen Rudin, December 17, 1973

1. **Introduction.** The purpose of this note is to announce a generalization of the Sz.-Nagy-Foiaş model theory for contractions to arbitrary bounded operators. We also indicate how invariant subspaces are described by this model theory.

The Russians, for example, Livšic [**14**] and Brodskiĭ and Livšic [**7**], have studied model theories for various classes of operators, often including some noncontractions. Recently there has been some work, for example, Davis and Foiaş [**13**], Brodskiĭ, Gohberg, and Kreĭn [**9**], and Brodskiĭ [**8**] on characteristic functions for noncontractions. Our work is closely related to that of Clark [**11**] and depends heavily for inspiration upon the canonical models of de Branges-Rovnyak [**5**].

In many of these papers, one of the main points is the connection between factorizations of the characteristic function B and invariant subspaces. Sz.-Nagy-Foiaş [**15**] found a precise condition on a factorization of B to insure that it results (for contractions) in an invariant subspace. Also the work of de Branges [**4**], [**5**] should be mentioned. Most recently Clark [**12**] has taken this problem up for invertible noncontractions. We propose to study this problem for the class of bounded noncontractions.

2. **Model theory.** The characteristic operator function $B(z)$ of a bounded Hilbert space operator T is defined by

$$B(z) = -TJ_T + z\,|I - TT^*|^{1/2}(I - zT^*)^{-1}\,|I - T^*T|^{1/2} \tag{1}$$

where $J_T = \operatorname{sgn}(I - T^*T)$ and where B acts from $\mathscr{D}_T$, the closure of the range of $|I-T^*T|^{1/2}$, to $\mathscr{D}_{T^*}$. A basic problem of model theory is to construct from B, in a canonical way, a bounded operator T such that B satisfies (1).

Let $\mathscr{C}_*$ and $\mathscr{C}$ be Hilbert spaces, and let $B(z):\mathscr{C}\to\mathscr{C}_*$ be analytic in a neighborhood D of 0. We also assume that D is symmetric about the real line. Let

$$J = \operatorname{sgn}(I - B(0)^*B(0)), \qquad J_* = \operatorname{sgn}(I - B(0)B(0)^*), \qquad \operatorname{sgn} 0 = 1.$$

AMS (MOS) subject classifications (1970). Primary 47A45.

Let $\bar{B}(z)=B(\bar{z})^*$. For each $c \in \mathscr{C}_*$, $d \in \mathscr{C}$, let

$$k(w, z)(c, d)$$
$$(2) \quad = ([J_* - B(z)JB(w)^*]c/(1 - z\bar{w}) + [B(z) - B(\bar{w})]d/(z - \bar{w}),$$
$$[\bar{B}(z) - \bar{B}(\bar{w})]Jc/(z - \bar{w}) + [J - \bar{B}(z)J_*\bar{B}(w)^*]c/(1 - z\bar{w}));$$

then $k(w, z)(c, d)$ is an ordered pair of functions each analytic in z on D, the first component taking values in $\mathscr{C}_*$, the second in $\mathscr{C}$. Let $H_0=\{k(w, z)(c, d)|w \in D, c \in \mathscr{C}_*, d \in \mathscr{C}\}$ and H_1 be all linear combinations of elements of H_0. For $k(w_1, z)(c_1, d_1)$ and $k(w_2, z)(c_2, d_2)$ two elements of H_0, define

$$(3) \qquad \langle k(w_1, z)(c_1, d_2), k(w_2, z)(c_2, d_2)\rangle = \langle k(w_1, w_2)(c_1, d_1), (c_2, d_2)\rangle,$$

the second inner product taken in $\mathscr{C}_* \times \mathscr{C}$, the elements of which are written as row vectors. Extend (3) to H_1 by linearity. It can be shown (see **[1]**) that a necessary condition that $B(z)$ be a characteristic function is that (3) be positive-definite.

When this is the case, we let $\mathscr{D}(B)$ (following notation of de Branges-Rovnyak) be the completion of the pre-Hilbert space H_1. The elements of $\mathscr{D}(B)$ can be taken to be of the form $(f(z), g(z))$, where both f and g are analytic on D, f is valued in C_*, g in $\mathscr{C}$. When convenient, we write (f, g) rather than $(f(z), g(z))$.

Define a linear operator $S: H_0 \to H_1$ by

$$S: k(w, z)(c, d) \to$$
$$\bar{w}^{-1}[k(w, z)(c, 0) - k(0, z)(c, 0)] + \bar{w}k(w, z)(0, d) - k(0, z)(J_*B(\bar{w})d, 0)$$

and extend S to H_1 by linearity. The following theorem (proved in **[1]**) shows that this construction yields a model for a general bounded operator.

THEOREM 1. *The operator S extends by continuity to a bounded operator (also S) on $\mathscr{D}$ (B), and B coincides with the characteristic function of S. A formula for S independent of a kernel function representation is*

$$(4) \quad \begin{aligned} &S: (f(z), g(z)) \to (zf(z) - B(z)Jg(0), [g(z) - g(0)]/z) \quad \textit{and} \\ &S^*: (f(z), g(z)) \to ([f(z) - f(0)]/z, zg(z) - \bar{B}(z)J_*f(0)). \end{aligned}$$

Also proved in **[1]** are the useful relations

$$(5) \qquad I - SS^* = e_1(0)^*J_*e_1(0), \qquad I - S^*S = e_2(0)^*Je_2(0),$$

where $e_1(0): \mathscr{D}(B) \to \mathscr{C}_*$ is defined by $(f, g) \to f(0)$ and $e_2(0): \mathscr{D}(B) \to \mathscr{C}$ is given by $(f, g) \to g(0)$. It follows from the definition of $\mathscr{D}(B)$ that

$$(6) \qquad e_1(0)^*c = k(0, z)(c, 0) \quad \text{and} \quad e_2(0)^*d = k(0, z)(0, d).$$

A consequence of Theorem 1 is that the positive-definiteness of the bilinear form (3) is necessary and sufficient for B to be a characteristic operator function. There results a new proof of the theorem of Brodskiĭ [**8**]. Clark [**11**] and Brodskiĭ, Gohberg and Kreĭn [**9**] handle the case where $B(z)$ is invertible in D.

3. **Invariant subspaces.** We restrict ourselves, for the purpose of studying invariant subspaces, to factorizations we call standard.

DEFINITION 1. For $B(z):\mathscr{C}_1\to\mathscr{C}_3$ a characteristic operator function, the factorization $B=B_2\cdot B_1$ $(B_1(z):\mathscr{C}_1\to\mathscr{C}_2,\ B_2(z):\mathscr{C}_2\to\mathscr{C}_3)$ is said to be standard if:

(i) B_2 and B_1 are also characteristic operator functions.

(ii) On $\mathscr{C}_1$, $J_1\equiv\operatorname{sgn} I-B_1(0)^*B_1(0)=\operatorname{sgn} I-B(0)^*B(0)$.
On $\mathscr{C}_2$, $J_2\equiv\operatorname{sgn} I-B_2(0)^*B_2(0)=\operatorname{sgn} I-B_1(0)B_1(0)^*$.
On $\mathscr{C}_3$, $J_3\equiv\operatorname{sgn} I-B_2(0)B_2(0)^*=\operatorname{sgn} I-B(0)B(0)^*$.

Let $[c,d]_i=\langle J_ic,d\rangle$ be the associated indefinite inner product on $\mathscr{C}_i$. Call an operator X: $\mathscr{C}_i\to\mathscr{C}_k$ J-contractive if $[Xc,Xc]_k\leqq[c,c]_i$, $i,k=1,2,3$. Then the condition for a factorization to be standard is essentially that $B(z)$ be the product of J-contractions. This generalizes the situation in the contraction case, where representations of B as a product of contractions is studied (Sz.-Nagy-Foiaş [**15**]).

If $B(z)$ is of the form (1), we show that standard factorizations correspond to invariant subspaces, if not for T, then for $T\oplus U$, where U is a unitary operator.

THEOREM 2. *Let $B=B_2\cdot B_1$ be a standard factorization. Then there is a partial isometry Γ from $\mathscr{D}(B_2)\oplus\mathscr{D}(B_1)$ onto $\mathscr{D}(B)$ given by*

$$\Gamma:(f_2,g_2)\oplus(f_1,g_1)\to(f_2+B_2f_1,\ \bar{B}_1g_2+g_1). \tag{7}$$

The difficulty for invariant subspaces is that Γ may have a nontrivial kernel. The situation is best understood by defining another space, the overlapping space of de Branges and Rovnyak.

DEFINITION 2. Let $B=B_2\cdot B_1$ be a standard factorization. Define a space $\mathscr{E}$ $(=\mathscr{E}(B_2\cdot B_1))$ by

$$\mathscr{E}=\{(f,g)\mid(B_2f,-J_2g)\in\mathscr{D}(B_2)\text{ and }(f,-\bar{B}_1J_2g)\in\mathscr{D}(B_1)\}$$

with a norm given by

$$\|(f,g)\|^2_{\mathscr{E}}=\|(B_2f,-J_2g)\|^2_{\mathscr{D}(B_2)}+\|(f_1,-\bar{B}_1J_1g)\|^2_{\mathscr{D}(B_1)}.$$

THEOREM 3. (i) *$\mathscr{E}$ is isometrically isomorphic to $\mathscr{N}$ = the kernel of Γ (see Theorem 2) under the map $\chi:(f,g)\to(B_2f,-J_2g)\oplus(-f,\bar{B}_1J_2g)$;*

(ii) *the operator U defined by*

$$U:(f(z),g(z))\to(zf(z)+g(0),\ [g(z)-g(0)]/z) \tag{8}$$

is unitary on $\mathscr{E}$ with adjoint

$$U^*:(f(z), g(z)) \to ([f(z) - f(0)]/z, zg(z) + f(0)).$$

Note that (i) follows directly from the definitions. The proof of (ii) is a direct computation, using relations (4) and (5) in the appropriate spaces.

It follows from (ii) and Theorem 1 of de Branges [**2**] that $\mathscr{E}$ is a space of the type $\mathscr{E}(\varphi)$ studied by de Branges and Rovnyak [**5**].

The above analysis gives rise to an invariant subspace theorem, known to the Russians in terms of a somewhat different model theory [**6**].

THEOREM 4. *Let $B=B_2 \cdot B_1$ be a standard factorization. Let S, S_1 and S_2 be the model operators in $\mathscr{D}(B)$, $\mathscr{D}(B_1)$ and $\mathscr{D}(B_2)$ respectively, and let U be the unitary operator of Theorem 3 in $\mathscr{E}(B_2 \cdot B_1)$. Then*

$$\text{(i)} \quad \begin{aligned} \Gamma' &= (f_2, g_2) \oplus (f_1, g_1) \\ &\to (f_2 + B_2 f_1, \bar{B}_1 g_2 + g_1) \oplus (\chi^{-1} P_{\mathscr{N}}(f_2, g_2) \oplus (f_1, g_1)) \end{aligned}$$

is unitary from $\mathscr{D}(B_2)\oplus\mathscr{D}(B_1)$ onto $\mathscr{D}(B)\oplus\mathscr{E}$, where $\mathscr{N}$ and χ are as in Theorem 3, (ii) *$\mathscr{M}=\Gamma'((0)\oplus\mathscr{D}(B_1))$ is an invariant subspace for $S\oplus U$; $S\oplus U|\mathscr{M}$ is unitarily equivalent to S_1 via Γ',* (iii) *$\mathscr{M}^\perp=\Gamma'(\mathscr{D}(B_2)\oplus(0))$ is invariant for $S^*\oplus U^*$; $S^*\oplus U^*|\mathscr{M}^\perp$ is unitarily equivalent to S_2^* via Γ'.*

We hope to publish details elsewhere.

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