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# A PRODUCT FORMULA FOR AN ARF-KERVAIRE INVARIANT

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In [1] we introduced an Arf-Kervaire type of invariant $\sigma(M) \in Z_8 = Z/8Z$ defined for closed compact, even-dimensional manifolds $M$ having a certain kind of orientation (see below). In this announcement we give a product formula for $\sigma$. Our results are applicable to Poincaré duality spaces, but for simplicity we give them for smooth manifolds. A special case of our formula was given in [2].

Let $v^m$ be the map

$$v^m = \prod v_i : BO_k \to \prod_{2i>m} K(Z_2, i),$$

where $v_i \in H^i(BO_k)$ is the $i$th Wu class. Let $BO_k^m$ be the fibration over $BO_k$ induced by $v^m$ from the contractible fibration. Let $\zeta_k$ be the universal $k$-plane bundle over $BO_k$, and let $\zeta_k^m = p^*\zeta_k$, where $p: BO_k^m \to BO_k$ is the projection. The Whitney sum map, $\zeta_k \times \zeta_l \to \zeta_{k+l}$, lifts to a map $\mu: \zeta_k^m \times \zeta_l^n \to \zeta_{k+l}^{m+n}$.

If $M$ is an $m$-manifold, a *Wu orientation* of $M$ is a bundle map $V: \nu \to \zeta_k^m$, where $\nu$ is the normal bundle of $M \subset R^{m+k}$. (Every manifold has a Wu orientation.) If $U$ and $V$ are Wu orientations on $M$ and $N$, $M \times N$ has a product orientation $U \times V$ defined in the obvious way. (For a detailed account of these ideas see [2].) Hereafter, manifold means a compact, closed, smooth manifold with a Wu orientation. $M \times N$ denotes the product manifold with the product orientation. The definition of $\sigma$ given in [1] is applicable to $M$, with its Wu orientation, if $\dim M = 2n$. Let $\sigma(M) = 0$ if $\dim M = 2n+1$. The definition of $\sigma$ in [1] depended on a choice $\lambda_n: \pi_{2n+k}(T(\zeta_k^{2n}) \wedge K(Z_2, n)) \to Z_4$. Choose such $\lambda_n$'s for each $n$ (such that $\lambda_n(\alpha_n) = 2$ in the notation of [1]). ($\lambda_{2n}$ can and should be chosen so that $\sigma(M) = \text{index}(M) \bmod 8$ if $M$ is an oriented (in the usual sense) $4n$-manifold.) Since we killed $v_{n+1}$ to form $BO_k^n$, $S^n$ has a nontrivial Wu orientation. Let $\bar{S}^n$ denote $S^n$ with this orientation. It turns out that

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$\sigma(\bar{S}^n \times M) \in \{0, 4\}$. Let $\sigma^n(M) \in Z_2$, $n>0$, be defined by $4\sigma^n(M) = \sigma(\bar{S}^n \times M)$, where $4: Z_2 \to Z_8$ is the inclusion.

THEOREM 1.1. *The maps* $\mu: \zeta_k^m \times \zeta_l^n \to \zeta_{k+l}^{m+n}$ *can be chosen so that the following formulas hold*:

$$\sigma(M \times N) = \sigma(M)\sigma(N) + \sum_n 4(\sigma^n(M)\sigma^n(N)),$$

$$\sigma^n(M \times N) = \sigma(M)\sigma^n(N) + \sigma^n(M)\sigma(N).$$

REMARK. $\sigma^n(M)=0$ for $n>\dim M$ or $n+\dim M$ odd. In the above formula, $\sigma(M)\sigma^n(N)$ means $(\sigma(M) \bmod 2)\sigma^n(N)$. In [1] it was shown that $\sigma(M)=$Euler characteristic of $M$ modulo 2.

THEOREM 1.2. *If* $m=\dim M$, $m-n$ *is even and* $v_{(m-n)/2}(M)=0$, *then* $\sigma^n(M)=0$; $\sigma^n(\bar{S}^n)=1$ $(v_0=1)$; $\sigma(\bar{S}^n \times \bar{S}^n)=4$; $\sigma^n(\bar{S}^n \times RP^{2m})=1$.

A somewhat more amusing way of writing Theorem 1.1 is as follows: Let $A=Z_8[a_n]$, $n=1, 2, \cdots$, modulo the relations $2a_n=a_na_m=0$, $n\neq m$, $a_n^2=4$. Let $\sum(M)=\sigma(M)+\sum \sigma^n(M)a_n$.

THEOREM 1.3. $\sum(M \times N)=\sum(M)\sum(N)$.

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