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ON NONLINEAR FREDHOLM OPERATOR EQUATIONS

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Let L denote a bounded linear Fredholm operator of index $p \geqq 0$ mapping a Hilbert space H into itself with $\dim \operatorname{Ker} L = d$ and $\dim \operatorname{coKer} L = d_*$. In this note, we consider the solvability of the operator equation

$$(1) \qquad Lu + Nu = f, \qquad f \in H,$$

where N is a C^1 *uniformly bounded* (nonlinear) mapping of H into itself. If $p=0$ and N is compact, the solvability of (1) is generally studied by means of the Leray-Schauder degree [**1**]. However for $p>0$, more general methods of study are required, since A. Švarc [**2**] showed the homotopy classes of (singularity-free) mappings $L+N$ restricted to the unit sphere in H are in (1-1) correspondence with the stable homotopy group $\pi_{n+p}(S^n)$ $(n>p+1)$. More recently, L. Nirenberg [**3**] studied criteria for solvability of coercive boundary value problems for elliptic systems (defined on bounded domains or compact manifolds) of the form (1) (with $N(u)$ satisfying additional compactness and asymptotic properties). He showed that the criteria could be expressed in terms of the nontriviality of the *stable* homotopy class of a certain continuous mapping μ of the $(d-1)$-sphere $S^{d-1} \subset \operatorname{Ker} L$ into the (d_*-1)-sphere $S^{d_*-1} \subset \operatorname{coKer} L$. Nirenberg remarks that this criterion is difficult to apply since, in general, it is not known how to compute the stable homotopy class of μ.

Here we take up this solvability problem from a simple Hilbert space point of view and sharpen the results just mentioned in several respects. First, we remove the compactness requirement for the operator N, thus allowing the applicability of our results to elliptic systems (that can be put in the form (1)) defined over unbounded domains and noncompact manifolds. More importantly, we derive a criterion for solvability of (1) based on the nontriviality of the *homotopy class* $[\eta]$ of a mapping η of $S^{d-1} \to S^{d_*-1}$ (analogous to the mapping μ of [**3**]), provided L "dominates" N in the sense described below. More generally, if L "dominates" N apart from a finite-dimensional subspace of dimension m (as always occurs

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in the cases described in [**3**]), our criterion is based on the nontriviality of the homotopy class of the mth iteration of the (Freudenthal) suspension homomorphism $E^m[\mu]$. In applications to elliptic systems, the integer m is found by relating the spectrum of L to the "size" of N. In case L is selfadjoint, we find a necessary and sufficient condition for the solvability of (1) that implies the openness of the range of $L+N$. Full details of the proofs will be given elsewhere.

1. **The solvability result.** In accord with [**3**], we make the following assumptions on the operator N, and its Fréchet derivative $N'(u)$:

(I) $\|N'(u)\| \leqq c$, where c is some constant independent of u;

(II) $\eta(a)=\lim_{r\to\infty} P^*\{N(ra+x)-f\}$ exists uniformly for $\|x\|$ uniformly bounded and $a \in \partial\Sigma = \operatorname{Ker} L \cap \{w \mid \|w\|=1\}$ where $\eta(a)\neq 0$. Here P^* is the canonical projection of H onto coKer L.

Under these assumptions we prove

THEOREM 1. *Suppose that apart from a finite-dimensional subspace $W=\operatorname{Ker} L \oplus V$ of H, L dominates N in the sense that*

$$(2) \qquad \|Lw\| \geqq (c+\varepsilon)\,\|w\| \quad \textit{for some fixed } \varepsilon > 0.$$

Then, if $\dim V = m$, *equation* (1) *is solvable provided the mth iterate $E^m[\tilde{\eta}]$ of the suspension homomorphism E of the homotopy class $[\tilde{\eta}]$ of $\tilde{\eta}=\eta/|\eta|$ is a nontrivial element of $\pi_{m+d-1}(S^{m+d_*-1})$. In particular,* (1) *is solvable provided $[\tilde{\eta}]$ is nontrivial if either $m=0$, $p=0$, or more generally E^m is an isomorphism of $\pi_{d-1}(S^{d_*-1})$ into $\pi_{m+d-1}(S^{m+d_*-1})$.*

SKETCH OF PROOF. We begin by replacing equation (1) by the system

$$(3) \quad \begin{aligned} &\text{(a)} \qquad L^*Lu + L^*Nu - L^*f = 0,\\ &\text{(b)} \qquad P^*(Nu - f) = 0. \end{aligned}$$

This pair implies that $Au = Lu+Nu-f \in \operatorname{Ker} L^* \cap [\operatorname{Ker} L^*]^\perp$, so that (1) is satisfied if (3) holds for some $u \in H$. The converse is immediate.

To study the solvability of (3a), we utilize hypothesis (I) to reduce (3a) to a finite-dimensional system. To this end, let $H=\operatorname{Ker} L \oplus V \oplus H_1$ where $H_1 = W^\perp$ and denote by P_1 the canonical projection of H onto H_1. Then, for $w \in H_1$ and arbitrary $x \in H$, (2) implies $(L^*A'(x)w, w) \geqq \varepsilon\|w\|^2$, where $A'(x)$ denotes the Fréchet derivative of $A(x)$. Consequently by Hadamard's theorem (see [**4**, pp. 16–18]), $P_1L^*A(x+w)$ is a global homeomorphism of H_1 onto itself. Now (3a) is automatically satisfied on $\operatorname{Ker} L$. Hence if a tentative solution of the system (3) is written $u=z+y+w$ with $z \in \operatorname{Ker} L$, $y \in V$, and $w \in H_1$, it suffices to solve the finite-dimensional system

$$(4) \qquad L^*Ly + P_v(L^*N(u) - f) = 0, \qquad P_v \equiv \operatorname{proj} H \to V.$$

Now we consider the finite-dimensional system (3b) and (4), and note that on a sufficiently large sphere in W, the mapping so defined is homotopic to the mapping

$$\Phi(y, z) = (y, P^*(N(z) - f)). \tag{5}$$

This fact follows since the uniform boundedness of $N(u)$ over H yields *uniform* a priori bounds for $\|w\|$ and $\|y\|$ satisfying (3). Now, let $\eta(a)$ be the mapping of $S^{d-1} \to \text{coKer}\, L$ defined in (II), and $\tilde{\eta}$ be the associated normalized mapping. Then we observe that the homotopy class $[\tilde{\Phi}]$ of the normalized mapping $\tilde{\Phi} = \Phi/|\Phi|$ regarded as a mapping from $S^{m+d-1} \to S^{m+d_*-1}$ satisfies $[\tilde{\Phi}] = E^m[\tilde{\eta}]$, where E^m is the mth iterate of the Freudenthal suspension homomorphism. Thus the theorem follows.

2. **Necessary and sufficient conditions for solvability.** As an application of Theorem 1 (in case L is selfadjoint), we prove

THEOREM 2. *Suppose L is selfadjoint, N satisfies the hypotheses of Theorem* 1, *and in addition, for all positive numbers r*

$$(N(u), a) < (\phi(a), a) \tag{6}$$

with

$$\phi(a) = \lim_{r \to \infty} P^*N(ra), \qquad a \in \partial\Sigma \quad \text{and} \quad x \perp \text{Ker}\, L.$$

Then, (a) *a necessary and sufficient condition for the solvability of* (1) *is*

$$(f, a) < (\phi(a), a) \quad \textit{for all } a \in \partial\Sigma; \tag{7}$$

and (b) *the mapping $L+N$ has open range.*

PROOF OF (a). The necessity of (7) follows immediately from (6) and the selfadjointness of L. The sufficiency of (7) follows from Theorem 1 and hypothesis (II), since in this case index $L=0$ and for r sufficiently large the Brouwer degree

$$\deg(PN(z), Pf, |z| < r) = \deg(\phi, f, |z| < r) = 1,$$

where P denotes the canonical projection of H onto Ker L.

PROOF OF (b). Let $f_0 \in \text{Range}(L+N)$. Then, to prove $L+N$ has open range, it suffices to show that $S_\varepsilon = \{f \mid \|f - f_0\|_H < \varepsilon\}$ (for some $\varepsilon > 0$) also lies in Range$(L+N)$. This last fact follows immediately from (7) for $f \in [\text{Ker}\, L]^\perp \cap S_\varepsilon$ for any ε and for $\|P\{f - f_0\}\|$ sufficiently small, by virtue of the strict inequality in (7) and the finiteness of dim Ker L.

REMARK. Theorems 1 and 2 are readily applicable to the study of boundary value problems of semilinear elliptic systems defined on manifolds or domains in Euclidean space $\boldsymbol{R}^N$. In such cases (2) of Theorem 1 will always be satisfied for some finite-dimensional subspace W provided the spectrum of the associated linear differential operator L is discrete.

To find relations between the solutions of (1), we prove

THEOREM 3. *Suppose that in addition to the hypotheses of Theorem* 2, N *is a completely continuous operator. Then, if* $f \in \text{Range}(L+N)$ (*except for a possible set of the first Baire category*) (i) *each solution* x *of* (1) *is nondegenerate* (*i.e.* $L+N'(x)$ *is an invertible linear operator*), (ii) *the solutions of* (1) *are finite in number, and* (iii) *if the Fréchet derivative of* N *is selfadjoint, and off* W, $L+N'>0$, *while* $N'<0$ *on* H, *the following Morse inequalities hold, where the* M_i *denote the number of solutions of* (1) *of Morse index* i *and* $n=\dim[Wn\{L\leqq 0\}]$:

$$
\begin{gathered}
M_n \geqq 1, \qquad M_{n+1} - M_n \geqq -1, \\
M_{n+2} - M_{n+1} + M_n \geqq 1, \cdots, \sum_{i=n}^{\dim W} (-1)^i M_i = \pm 1.
\end{gathered} \tag{8}
$$

SKETCH OF PROOF. We first note that the solvability of (1) implies (by virtue of Theorem 2) (*) the uniform boundedness of any sequence $\{u_n\}$ for which $\|Lu_n+Nu_n-f\|\to 0$. This implies (by the inverse function theorem and Smale's version of Sard's theorem) that, apart from a possible exceptional set of first Baire category, the solutions of (1) are nondegenerate and isolated. The complete continuity of N and the Fredholm property of L yield the finiteness of these solutions. Moreover, by virtue of (*), the functional $I(u)$ defined by the relation $I'(u)=Lu+Nu-f$ satisfies the Palais-Smale condition (C) on H. Consequently, a modification of Smale's version of Morse theory on Hilbert space establishes the relations (8).

ADDED IN PROOF. Theorem 1 is true for mappings $L+N$ of a Banach space X into a Banach space Y provided hypothesis (I) is altered to read $\|P_1N'(u)\|\leqq c$ where P_1 is a projection of Y onto $L(X/W)$. Moreover, examples show that Theorem 1 is sharp in the senses that (i) the homotopy class $[\tilde{\eta}]$ may be nontrivial and equation not solvable, (ii) the stable homotopy class of $\tilde{\eta}$ may be trivial, yet equation (1) is solvable by the criterion given in Theorem 1 with $m\neq 0$.

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