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THE ORIENTED TOPOLOGICAL AND PL COBORDISM RINGS[1]

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1. **Introduction and statement of results.** In this note we announce results on the 2-local structure of the oriented topological cobordism ring Ω_*^{TOP} and its PL analogue Ω_*^{PL}.

It is a well-known consequence of transversality that

$$\Omega_*^{\mathrm{TOP}} = \pi_*(\mathrm{MSTOP}), \qquad * \neq 4 \quad \text{and} \quad \Omega_*^{\mathrm{PL}} = \pi_*(\mathrm{MSPL}),$$

where MSTOP and MSPL are the oriented Thom spectra.

Also, the homotopy theory of these spectra divides into two distinct problems: the theory *at* the prime 2 and the theory *away* from 2. We let $\boldsymbol{Z}_{(2)}$ denote the integers localized at 2 and $\boldsymbol{Z}[\frac{1}{2}]$ the integers localized away from 2.

Sullivan [9] showed that the free part of $\Omega_*^{\mathrm{TOP}} \otimes \boldsymbol{Z}[\frac{1}{2}]$ $(=\Omega_*^{\mathrm{PL}} \otimes \boldsymbol{Z}[\frac{1}{2}])$; $\Omega_*^{\mathrm{TOP}}/\mathrm{Tor} \otimes \boldsymbol{Z}[\frac{1}{2}]$ is a polynomial algebra with one generator in each dimension congruent to zero mod 4.

At the prime 2 Browder, Liulevicius and Peterson [2] show that the localized spectra $\mathrm{MSTOP}_{(2)}$ and $\mathrm{MSPL}_{(2)}$ become wedges of Eilenberg-Mac Lane spectra. Hence the homotopy theory is a direct consequence of the homology theory. In particular,

1.1 $$(\Omega_*^{\mathrm{TOP}}/\mathrm{Tor}) \otimes \boldsymbol{Z}_{(2)} = H_*(\mathrm{BSTOP}; \boldsymbol{Z}_{(2)})/\mathrm{Tor}$$

and similarly in the PL case.

Let M_0^{4n}, $n>1$, be the Milnor manifold of index 8 constructed by plumbing disk tangent bundles of S^{2n} (see Browder [1, p. 122]). The boundary of M_0^{4n} is the PL sphere S^{4n-1}. We set $M^{4n} = M_0^{4n} \cup_\partial CS^{4n-1}$ to obtain a closed PL manifold of index 8.

In the rest of this note, $P(X)$, $E(X)$ and $\Gamma(X)$ will denote the polynomial algebra, exterior algebra, and divided power algebra, respectively generated by the set X. For a natural number n, $\alpha(n)$ will be the number of nonzero terms in the dyadic expansion and $\nu(n)$ the 2-adic valuation ($n = 2^{\nu(n)}$ odd).

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THEOREM A. *As rings*,

$$(\Omega_*^{\mathrm{TOP}}/\mathrm{Tor}) \otimes Z_{(2)}$$
$$= P\{[CP^{2n}] \mid \alpha(n) < \nu(n) + 4\} \otimes \Gamma\{[M^{4n}] \mid \alpha(n) \geqq \nu(n) + 4\}.$$

Moreover, $(\Omega_*^{\mathrm{PL}}/\mathrm{Tor})\otimes Z_{(2)} = (\Omega_*^{\mathrm{TOP}}/\mathrm{Tor})\otimes Z_{(2)}$. *Here* CP^{2n} *is the complex projective space.*

The torsion structures of $\Omega_*^{\mathrm{TOP}}\otimes Z_{(2)}$, $*\neq 4$ and $\Omega_*^{\mathrm{PL}}\otimes Z_{(2)}$ are very involved, and even though our techniques give the groups, we know comparatively little about the explicit generators. However, there are a finite number of explicit constructions—twisted products, and Massey products—which generate the torsion from a small set of "basic" torsion manifolds. Among these generators are specific ones given by relations among the Milnor manifolds and the CP^{2n}'s. For example, the relation below (the first which occurs) generates a $Z/2Z$ direct summand in Ω_8^{PL}.

1.2 $$2\{7[M^8] - 200[CP^2 \times CP^2] + 144[CP^4]\} = 0$$

while in dimension 12 there is a $Z/4Z$ summand generated by the relation

1.3 $$4\{31[M^{12}] - 1620[CP^6] + 5292[CP^4]\cdot[CP^2] - 3920[CP^2]^3\} = 0.$$

1.2 and 1.3 are a little surprising since it is well known that the smallest multiple of M^8 which is actually PL homeomorphic to a differentiable manifold is $28M^8$ while the corresponding number for M^{12} is 992.

In the rest of this note, all spaces and maps are to be taken in the 2-local category (see **[10]** for a precise definition). Unless otherwise indicated $H_*(X)$ ($H^*(X)$) will denote homology (cohomology) of X with Z coefficients. (*Note.* $H_*(X; Z) = H_*(X; Z_{(2)})$ when X is 2-local.)

2. **Preliminaries.** *The map* $\mathrm{BSG}\to B(\mathrm{G/TOP})$. It is a well-known result of Sullivan that G/TOP is a product of Eilenberg-Mac Lane spaces. In **[7]** and **[8]** specific homotopy equivalences

$$K: G/\mathrm{TOP} \to \prod_{n\geqq 1} K(Z_{(2)}, 4n) \times K(Z/2, 4n-2)$$

were constructed. The mapping K depends on the "genus" used in the "surgery formulas". In this note we use the map defined in **[7]**.

In **[6]** we examined the space $B(G/\mathrm{TOP})$ as well as the natural map $B\pi: \mathrm{BSG}\to B(G/\mathrm{TOP})$. The main result there is

PROPOSITION 2.1. (i) *There is an H-map*

$$BK: B(G/\mathrm{TOP}) \to \prod_{n\geqq 1} K(Z_{(2)}, 4n+1) \times K(Z/2, 4n-1)$$

with $\Omega(BK \circ B\pi) = K \circ \pi$ *and* BK *a homotopy equivalence* ($\pi: SG \to G/\mathrm{TOP}$ *the natural map*).

(ii) *The class* $B\pi^*(K_{4n+1})$ *is divisible by precisely* $2^{\alpha(n)-1}$, *where* $K_{4n+1} = (BK)^*$ (*fundamental class*).

Next we specify the classes $(B\pi)^* K_{4n+1}$ more precisely. To do this we will specify the structure of the $Z_{(2)}$ cohomology of BSG by determining its Bochstein spectral sequence (BSS). We first introduce 3 (acyclic) DG-Hopf algebras over $Z_{(2)}$ which will be our basic building blocks.

(I)
$$A_0\langle k\rangle = P\{p_n \mid n \geqq 1\} \otimes E\{e_n \mid n \geqq 1\},$$
$$\deg(p_n) = 4n, \qquad \deg(e_n) = 4n+1, \qquad \psi(p_n) = \sum p_i \otimes p_{n-i},$$
$$\psi(e_n) = \sum p_i \otimes e_{n-i} + e_i \otimes p_{n-i}, \qquad \delta(p_n) = 2^k e_n.$$

(II)
$$A_1\{x \mid k\} = P\{x\} \otimes E\{y\},$$
$$\deg x = 4n, \qquad \deg y = 4n+1, \qquad \psi(x) = 1 \otimes x + x \otimes 1,$$
$$\psi(y) = 1 \otimes y + y \otimes 1, \qquad \delta x = 2^k y.$$

(III)
$$A_2\{x \mid k\} = E\{y\} \otimes \Gamma\{x\},$$
$$\deg x = 4n, \qquad \deg y = 4n-1, \qquad \psi(y) = 1 \otimes y + y \otimes 1$$
and
$$\psi(x) = 1 \otimes x + x \otimes 1, \qquad \delta y = 2^k x$$
(hence $\delta(y \cdot \gamma_{2^r-1}(x)) = 2^{k+r}\gamma_{2^r}(x)$). If X is a graded set concentrated in degrees congruent to zero mod 4, we write $A_i\{X|k\} = \otimes_{x\in X} A_i\{x|k\}$, $i=1, 2$. Each of the DG-Hopf algebras above have an associated Bochstein spectral sequence $\{E_r(\ ), d_r\}$. From **[5]** we quote

PROPOSITION 2.2. *For* $r \geqq 2$, *the cohomology* BSS *of the space* BSG *is*
$$E_r(\mathrm{BSG}) = E_r(A_0\langle 3\rangle) \otimes E_r(A_2\{X \mid 2\})$$
for a suitable graded set X.

Let $j_r: H^*(\mathrm{BSG}) \to E_r(\mathrm{BSG})$ denote the natural reduction map. From **[3]** and **[6]** we have

PROPOSITION 2.3. (i) $j_3(2^{1-\alpha(n)} B\pi^*(K_{4n+1})) = e_n +$ *decomposable terms.*

(ii) $B\pi^*(K_{4n-1}) = 0$ *for* $\alpha(n) > 1$.

(iii) $\mathrm{Sq}^2 B\pi^*(K_{2^i-1}) = e_{2^i+1}$.

3. **The** DG**-Hopf algebra** $\mathscr{T}$**.** In §4 we show that the following DG-Hopf algebra over $Z_{(2)}$ is a split subalgebra of the BSS for BSTOP.

$$\mathscr{T} = P\{p_n \mid n \geqq 1\} \otimes P\{k_n \mid n \geqq 1\} \otimes E\{\varepsilon_n \mid n \geqq 1\},$$

$$\deg p_n = 4n, \qquad \deg k_n = 4n \quad \text{and} \quad \deg \varepsilon_n = 4n + 1,$$

$$\psi(p_n) = \sum p_i \otimes p_{n-i},$$

$$\psi(k_n) = 1 \otimes k_n + k_n \otimes 1, \qquad \psi(\varepsilon_n) = 1 \otimes \varepsilon_n + \varepsilon_n \otimes 1,$$

with differential structure given by

$$\delta p_n = 16 e_n, \qquad \delta k_n = 2^{\alpha(n)} \varepsilon_n \quad \text{where } e_n = \sum \varepsilon_i p_{n-i}.$$

Husemoller [4] has introduced a splitting of the Hopf algebra $P\{p_n \mid n \geqq 1\}$ as a tensor product of "smaller" Hopf algebras,

$$P\{p_n \mid n \geqq 1\} = \bigotimes_{n \text{ odd}} P\{P_{n,0}, p_{n,1}, \cdots, p_{n,i}, \cdots\}$$

($\deg p_{n,i} = 2^{i+2} n$). We split $\mathscr{T}$ accordingly,

$$\mathscr{T} = \bigotimes_{n \text{ odd}} \mathscr{T}(n),$$

$$\mathscr{T}(n) = P\{p_{n,0}, p_{n,1}, \cdots\} \otimes P\{k_{n,0}, k_{n,1}, \cdots\} \otimes E\{\varepsilon_{n,0}, \varepsilon_{n,1}, \cdots\}.$$

Here $k_{n,i} = k_{2^i n}$, $\varepsilon_{n,i} = \varepsilon_{2^i n}$ and the differential structure is (inductively) determined by

$$\delta(k_{n,i}) = 2^{\alpha(n)} \varepsilon_{n,i} \quad \text{and} \quad \delta(2^i p_{n,i} + \cdots + p_{n,0}^{2^i}) = 2^{i+4} \varepsilon_{n,i}.$$

LEMMA 3.1. (i) *If* $\alpha(n) < 4$, *then*

$$E_s(\mathscr{T}(n)) = P\{p_{n,0}, p_{n,1}, \cdots\} \otimes E_s(A_1\{k_{n,0}, k_{n,1}, \cdots \mid \alpha(n)\}).$$

(ii) *If* $\alpha(n) \geqq 4$, *then for* $s \geqq \alpha(n)$,

$$E_s(\mathscr{T}(n)) = P\{k_{n,0}, \cdots, k_{n,r-1}, k_{n,r} + p_{n,0}^{2^r}, p_{n,0}^{2^{r+1}}, p_{n,1}^{2^{r+1}}, \cdots\} \otimes E_s(A_1\{\bar{k}_{n,r}, \bar{k}_{n,r+1}, \cdots \mid \alpha(n)\}),$$

where

$$r = \alpha(n) - 4 \quad \textit{and} \quad \bar{k}_{n,r+i} = p_{n,i}^{2^r} + \sum_{j=1}^{i-1} p_{n,i-j-1}^{2^{r+j+1} - 2^{r+1}} \bar{k}_{n,r+i-j} + k_{n,r+i}.$$

4. **Theorem** A. There is a natural map BSO $\times G/$TOP$\to$BSTOP which on homology leads to

4.1 $$P\{a_n \mid n \geqq 1\} \otimes \Gamma\{b_n \mid n \geqq 1\} \xrightarrow{r_*} H_*(\text{BSTOP})/\text{Tor},$$

where a_n is dual to $p_1^n \in H^{4n}(\text{BSO})/\text{Tor}$ and b_n is spherical. We observe that the structure of $H_*(\text{BSTOP})/\text{Tor}$ follows at once if we can prove that $(H^*(\text{BSTOP})/\text{Tor}) \otimes Z/2 = E_\infty(\mathscr{T})$, where $E_\infty(\mathscr{T}) = \otimes_{n \text{ odd}} E_\infty(\mathscr{T}(n))$ is described in 3.1. Therefore the thrust of the argument is to evaluate the BSS of BSTOP.

Our starting point is the fibration sequence, $\cdots \to \mathrm{BSTOP} \to \mathrm{BSG} \to B(G/\mathrm{TOP}) \to \cdots$. It is convenient to decompose this sequence in two steps. Let

$$BK_1 = \prod_{i>1} K(\mathbf{Z}/2, 2^i - 1)$$

and

$$BK_2 = \prod_{n>1} K(\mathbf{Z}_{(2)}, 4n + 1) \times \prod_{\alpha(n)>1} K(\mathbf{Z}/2, 4n - 1).$$

We have the fibration sequences ($\Omega BK_i = K_i$)

$$\text{4.2} \qquad \begin{array}{l} \cdots \to K_1 \to BX \to \mathrm{BSG} \to BK_1 \to \cdots \\ \cdots \to K_2 \to \mathrm{BSTOP} \to BX \to BK_2 \to \cdots . \end{array}$$

LEMMA 4.3. (i) *There are graded sets X_1 and X_2 such that for $r \geqq 2$ the rth term in the BSS of BX is*

$$E_r(BX) = E_r(A_0\langle 4\rangle) \otimes E_r(A_1\{X_1 \mid 2\}) \otimes E_r(A_2\{X_2 \mid 2\}).$$

(ii) *The inclusion $i: K_1 \to BX$ maps $E_r(A_1\{X_1|2\})$ injectively into BSS for K_1.*

It follows from 2.5 and 4.3 above that

$$H^*(\mathrm{BSTOP}; \mathbf{Z}/2) = H^*(BX; \mathbf{Z}/2) \otimes H^*(K_2).$$

Let $j: K_2 \to \mathrm{BSTOP}$ be the map in 4.2. Our main technical result is

THEOREM 4.4. (i) *There are graded sets Y_1 and Y_2 such that for $r \geqq 2$*

$$E_r(\mathrm{BSTOP}) = E_r(\mathscr{T}) \otimes E_r(A_1\{Y_1 \mid 2\}) \otimes E_r(A_2\{Y_2 \mid 2\}).$$

(ii) *j^* maps $E_r(A_1\{Y_1|2\})$ monomorphically to the BSS for $\prod K(\mathbf{Z}_{(2)}; 4n) \times \prod_{\alpha(n)>1} K(\mathbf{Z}/2; 4n-2)$.*

We first give an exact sequence of spectral sequences,

$$\mathbf{Z}/2 \to E_r(A_1\{Y_1 \mid 2\}) \otimes E_r(A_2\{Y_2 \mid 2\}) \to E_r(\mathrm{BSTOP}) \to \hat{E}_r \to \mathbf{Z}/2,$$

satisfying (ii) and with $\hat{E}_2 = E_2(\mathscr{T})$. From dimensional considerations and because $j^*(k_n)$ is an infinite cycle and $j^*(p_n) = 0$, it follows that this sequence splits:

$$E_r(\mathrm{BSTOP}) = \hat{E}_r \otimes E_r(A_1\{Y_1 \mid 2\}) \otimes E_r(A_2\{Y_2 \mid 2\}).$$

Algebraic considerations lead to the pleasant fact that $\hat{E}_\infty$ is a polynomial algebra with one generator in each degree congruent to zero mod 4.

Since

$$\hat{E}_\infty = E_\infty(\mathrm{BSTOP}) = H^*(\mathrm{BSTOP})/\mathrm{Tor} \otimes \mathbf{Z}/2$$

we see that $H^*(\mathrm{BSTOP})/\mathrm{Tor}$ is a polynomial algebra. In particular the $4n$-dimensional primitives of $H_*(\mathrm{BSTOP})/\mathrm{Tor}$ are a copy of $\mathbf{Z}_{(2)}$.

We now employ a result of Morgan and Sullivan [**8**]. They construct a class $L_n \in H^{4n}(\mathrm{BSTOP})$ whose rational reduction is the (inverse) Hirzebruch class when restricted to $H^{4n}(\mathrm{BSO}:Q)$ and whose restriction to G/TOP is 8 ("surgery class"). Since the coefficient of p_n in the Hirzebruch class is $2^{\alpha(n)-1}$ (odd), it follows that

$$2^{\alpha(n)-1} \cdot \tau_*(b_n) = 8 \cdot \tau_*(s_n(a_1, \cdots, a_n)).$$

(s_n is the Newton polynomial.)

This equation implies that $\tau_*(\gamma_2 i(b_n))$ is divisible by 2 unless $\alpha(n) \geqq 4+\nu(n)$, and from this one can inductively conclude that

$$\hat{E}_r = E_r(\mathscr{T}).$$

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