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LEBESGUE SPACES FOR BILINEAR VECTOR INTEGRATION THEORY

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In this note we shall announce results concerning the structure of $L_E^1(m)$, the space of E-valued functions integrable with respect to a measure $m:\Sigma\to L(E, F)$, where $L(E, F)$ is the class of bounded operators from the Banach space E into the Banach space F. The bilinear integration theory introduced here is more restrictive than the one developed by Bartle **[1]**, but it is general enough to allow a norm to be defined on the integrable functions and to permit the study of weak compactness and convergence theorems; moreover, $L_E^1(m)$ lends itself in a natural way to the study of continuous operators $T:C_E(S)\to F$, where the domain is the space of continuous E-valued functions defined on the compact Hausdorff space S as follows: By Dinculeanu's representation theorem **[6]**, there exists a unique regular finitely-additive measure $m:\Sigma\to L(E, F^{**})$, where Σ is the family of Borel subsets of S, such that $T(f)=\int f\,dm$. If T is a weakly compact operator, Brooks and Lewis **[2]** have shown that m is countably additive, with range in $L(E, F)$. In addition, the set $N=\{|m_z|:z\in F_1^*\}$ is relatively weakly compact in ca(Σ)—here m_z is the E^*-valued measure defined by $m_z(A)e=\langle m(A)e, z\rangle$, and $|m_z|$ is the total variation function of m_z. Conversely, if N has the above property and E is reflexive, then T is weakly compact. A natural question is whether a Lebesgue space $L_E^1(m)\supset C_E(S)$ of m-integrable functions can be defined. If so, what convergence theorems can be proved, and how are the weakly compact sets characterized?

The setting is as follows. Let Σ be a σ-algebra of subsets of a set T, and $m:\Sigma\to L(E, F)$, a countably additive measure be given such that m is strongly bounded, that is, $\tilde{m}_{E,F}(A_i)\to 0$, whenever (A_i) is a disjoint sequence of sets ($\tilde{m}_{E,F}$ is the semivariation of m with respect to E and F **[6]**). It follows that $N=\{|m_z|:z\in F_1^*\}$ is relatively weakly compact in ca(Σ). Let λ be a positive control measure for m such that $\lambda\leqq\tilde{m}_{E,F}$ and

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$m \ll \lambda$. The set $F_E(N)$ is the set of λ-measurable E-valued functions f such that

$$N(f) = \sup\left\{\int |f|\, d\,|m_z| : z \in F_1^*\right\} < \infty.$$

For every $f \in F_E(N)$, define the integral $\int f\, dm \in F^{**}$ by $\langle z, \int f\, dm\rangle = \int f\, dm_z$, for $z \in F^*$. The closure of the E-valued simple functions in $F_E(N)$ is defined to be $L_E^1(m)$. By $L^1(m)$ we denote the Banach space of scalar m-integrable functions. The Vitali convergence theorem and the Lebesgue dominated convergence theorem are valid in $L_E^1(m)$, but the Beppo Levi theorem fails in general. We say that m has the Beppo Levi property if every increasing sequence of positive simple functions f_n, with $\sup_n N(f_n) < \infty$, is a Cauchy sequence in $L^1(m)$. In Theorem 1 below, it is seen that this crucial property is satisfied under mild conditions. In the case E is the scalar field, $L^1(m)$ includes the Bartle–Dunford–Schwartz integral as defined in [7, Chapter IV].

The authors have developed a more general theory in which Lebesgue spaces $L_E^1(N)$ are studied, where N is a family of positive measures not necessarily arising from a vector measure as above. The theory of associate spaces is introduced and representations of operators $T: L_E^1(N) \to F$ are given. To cover the case when S is a locally compact space, we construct the theory on δ-rings by means of localizable control measures. In this announcement we shall restrict ourselves to the above special case and a few representative theorems will be stated. The complete version of these results will appear in [4].

THEOREM 1. *If $\int f\, dm \in F$ for every $f \in F_E(N)$, then m has the Beppo Levi property. In particular, if F is weakly sequentially complete, then m has the Beppo Levi property.*

INDICATION OF THE PROOF. One can show that it is sufficient to prove that m has the Beppo Levi property if whenever $f \in F_E(N)$ and $A_n \searrow \varnothing$, then $N(f\chi_{A_n}) \to 0$. If we deny this, there exist $z_n \in F_1^*$ such that $\int_{A_n} |f|\, d|m_{z_n}| > \varepsilon > 0$ for every n, for some ε. By proving that

$$\int_{A_n} |f|\, d\,|m_{z_n}| = \sup\left\{\left|\int_{A_n} g\, dm_{z_n}\right|, g \text{ is simple, } |g| \leqq |f|\right\},$$

we may assume: $(*)\ |\int_{A_n} f_n\, dm_{z_n}| > \varepsilon, n = 1, 2, \cdots$, where the f_n are simple E-valued functions. Form $L_{E_0}^1(\Sigma_0, m)$, where Σ_0 is a separable σ-algebra containing (A_n) such that f and f_n are Σ_0-measurable; E_0 is the closure of the span of $f(T) \cup \bigcup_{n \geq 1} f_n(T)$. Let $F_0 \subset F$ be the separable Banach space generated by $\{\int h\, dm : h \in L_{E_0}^1(\Sigma_0, m)\}$. By using the Lebesgue dominated convergence theorem in the spaces $L_{E_0}^1(|m_z|)$, we see that $h \in F_{E_0}(N)$

implies $\int h\, dm \in F_0$, whenever h is Σ_0-measurable. By a diagonal process, assume that (z_n) converges on a countable dense subset of F_0; hence (z_n) converges on F_0. Let $\tau=\Sigma 2^{-n}|m_{z_n}|$, and define $A(\Sigma_0, f)$ to be the space of Σ_0-measurable functions $h: T\to E_0$ such that $|h|\leqq|f|$ a.e. τ; thus $A(\Sigma_0, f)$ is a complete subset of $L^1_{E_0}(\Sigma_0, \tau)$. Define $T_n: L^1_{E_0}(\Sigma_0, \tau)\to F_0$ by $T_n(h)=\int h\, dm_{z_n}$; note that $\|T_n\|\leqq 2^n$. It follows that $\lim T_n(h)$ exists for every $h\in A(\Sigma_0, f)$. Using the Baire category theorem, we deduce that (T_n) is equicontinuous at zero on $A(\Sigma_0, f)$. However, since $\int_{A_n}|f_n|\, d\tau\leqq\int_{A_n}|f|\, d\tau\to 0$, we have $\lim_n \int_{A_n} f_n\, dm_{z_k}=0$ uniformly in k. This contradicts $(*)$.

Using the above theorem, we are able to give sufficient conditions for a set to be relatively weakly compact in $L^1_E(m)$.

THEOREM 2. *Assume that E is reflexive and F is weakly sequentially complete. Suppose $K\subset L^1_E(m)$ is a set satisfying:*

(1) *K is bounded;*

(2) *$N(f\chi_{A_n})\to 0$ uniformly for $f\in K$, whenever $A_n\searrow\varnothing$.*

Then K is relatively weakly compact.

INDICATION OF THE PROOF. By the Eberlain-Smulian theorem, we may assume that K is a sequence of functions f_n. Let λ be a bounded control measure for m; hence $\lambda(A)\leqq N(\chi_A)$, for $A\in\Sigma$, and $N\ll\lambda$ uniformly. We assert that $\lim_{N(A)\to 0} N(f\chi_A)=0$ uniformly for $f\in K$. By hypothesis the set $Q=\{\int |f|\, d|m_z| : z\in F_1^*\}$ is uniformly countably additive. Also each measure in Q is absolutely continuous with respect to λ. By [**3**, Theorem 2.1] $Q\ll\lambda$ uniformly. This implies the above assertion. Since $\lambda\leqq N$, we see that K is a bounded subset of $L^1_E(\lambda)$; moreover, $\lim_{\lambda(A)\to 0}\int_A |f|\, d\lambda=0$ uniformly for $f\in K$. Hence K is relatively weakly compact in $L^1_E(\lambda)$ [**5**], [**3**]. Assume that (f_n) converges weakly to $f_0\in L^1_E(\lambda)$; we shall show that (f_n) converges weakly to f_0 in $L^1_E(m)$. It can be shown that $L^1_E(m)^*$ consists of E^*-valued measures of finite total variation, absolutely continuous with respect to λ. Let $\sigma\in L^1_E(m)^*$. Using the Phillips theorem, we prove that there exists a $g\in L^1_{E^*}(m)$ such that $\sigma(h)=\int gh\, d\lambda$, for $h\in L^1_E(m)$. The sequence (gf_n) is bounded in $L^1(\lambda)$, since $\|gf_n\|_1\leqq\|\sigma\| N(f_n)$; note also that $\lim_{\lambda(A)\to 0}\int_A |gf_n|\, d\lambda=0$ uniformly in n. As a result, (gf_n) is relatively weakly compact in $L^1(\lambda)$. Thus we may assume that (gf_n) converges weakly to $h\in L^1(\lambda)$. Let (B_k) be a sequence of sets with union equal to T such that g is bounded on each B_n. For fixed k, $g\chi_C\in L^\infty_{E^*}(\lambda)=L^1_E(\lambda)^*$, for every $C\in\Sigma\cap B_K$. Thus $\int_C gf_n\, d\lambda\to\int_C gf_0\, d\lambda$ and $\int_C f_n g\, d\lambda\to\int_C h\, d\lambda$. Consequently $gf_0=h$ a.e. λ on B_k, hence $gf_0=h$ a.e. λ on T. Therefore: (#) $gf_0\in L^1(\lambda)$ for every $g\in L^1_E(m)^*$. To show that $f_0\in L^1_E(m)$ we proceed in two stages. First of all, we define the "associate space" $F_{E^*}(N')$ of $F_E(N)$ to be the space of λ-measurable functions $g:T\to E^*$ such that $gf\in L^1(\lambda)$ for every $h\in L^1_E(m)$. Then we show that when m has the Beppo Levi property (which, by Theorem 1, it does in this instance), then $f_0\in L^1_E(m)$ if and only if $gf_0\in L^1(\lambda)$ for every $g\in F_{E^*}(N')$. Then we prove that $F_{E^*}(N')=L^1_E(m)^*$. The lengthy details are omitted. This, in conjunction with (#), implies that $f_0\in L^1_E(m)$.

COROLLARY 1. *Let F be weakly sequentially complete. Suppose* $(f_n)_{n\geq 0}\subset L^1(m)$. *If* $\int_A f_n\,dm\to\int_A f_0\,dm$ *for every* $A\in\Sigma$, *then* $f_n\to f_0$ *weakly in* $L^1(m)$.

From the construction in the proof of Theorem 2, we deduce the following corollary.

COROLLARY 2. *If E is reflexive and F is weakly sequentially complete, then* $L^1_E(m)$ *is weakly sequentially complete.*

REFERENCES

1. R. G. Bartle, *A general bilinear vector integral*, Studia Math. **15** (1956), 337–352. MR **18**, 289.

2. J. K. Brooks and P. W. Lewis, *Operators on function spaces*, Bull. Amer. Math. Soc. **78** (1972), 697–701. MR **45** #7494.

3. J. K. Brooks and N. Dinculeanu, *Strong additivity, absolute continuity and compactness in spaces of measures*, J. Math. Anal. Appl. **45** (1974), 156–175.

4. ———, *Lebesgue-type spaces for vector integration. Linear operators, weak completeness and weak compactness*, J. Math. Anal. Appl. (to appear).

5. S. D. Chatterji, *Weak convergence in certain special Banach spaces*, MRC Technical Sum. Report #443, University of Wisconsin, Madison, Wis., 1963.

6. N. Dinculeanu, *Vector measures*, Internat. Ser. of Monographs in Pure and Appl. Math., vol. 95, Pergamon Press, Oxford, 1967. MR **34** #6011b.

7. N. Dunford and J. T. Schwartz, *Linear operators*. I: *General theory*, Pure and Appl. Math., vol. 7, Interscience, New York, 1958. MR **22** #8302.

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