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TILING SPACE BY CONGRUENT POLYHEDRA

BY S. K. STEIN

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In §18 of his turn-of-the-century address, *Mathematical problems* [**1**, p. 467], Hilbert asked, "*Whether polyhedra also exist which do not appear as fundamental regions by groups of motions, by means of which nevertheless by a suitable juxtaposition of congruent copies a complete filling up of all space is possible*?" In 1928 Reinhardt [**2**] constructed in R^3 a polyhedron whose existence showed that the answer is "yes". This polyhedron is not a star body and there does not seem to be any way to modify his construction to obtain a star body.

R. Bambah and C. A. Rogers have pointed out to me that a construction I used in [**3**] readily supplies a star body example in R^5 for Hilbert's problem.

Let $K \subset R^5$ consist of a unit cube together with five arms, each composed of three unit cubes attached at nonopposite facets of the central cube. The sixteen cubes are parallel. (Centers of the cubes may be taken to be the origin together with the fifteen points $(i, 0, 0, 0, 0), \cdots, (0, 0, 0, 0, i)$, $i=1, 2, 3$.) It is shown in [**3**] that translates of K by integer vectors tile R^5 but not in a lattice manner.

Now modify K by shallow pyramidal dents and matching bumps to obtain a star body K' with the property that if a family of translates of K' (including K' itself) tile R^5, then the translation vectors must have only integer coordinates. Moreover, by making the dents and bumps of a

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different shape in each of the five axial directions, we can assume that in a tiling of R^5 by a family of congruent copies of K', all these copies are translates of K'.

Consequently K' tiles R^5, but not by a group of motions. Whether a star body that meets Hilbert's demands exists in R^3 or R^4 remains open.

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