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A CONDITIONAL LOCAL LIMIT THEOREM AND ITS APPLICATION TO RANDOM WALK

BY W. D. KAIGH

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1. **Introduction.** Let $X_1, X_2, \cdots$ be a sequence of i.i.d. (independent and identically distributed) random variables defined on a probability space (Ω, F, P). In all that follows we assume that the X_i are distributed on the lattice of integers with $EX_i=0$ and $EX_i^2=\sigma^2<\infty$. For the recurrent random walk S_n with $S_0=0$ and $S_n=X_1+\cdots+X_n$ for $n\geqq 1$, define the stopping time T either to be the first time S_n returns to zero or to be $+\infty$ if no such n exists. We shall assume further that the random walk S_n is aperiodic. It is well known that T is finite with probability one and that $n^{1/2}P\ [T>n]$ converges to the limit $(2/\pi)^{1/2}\sigma$ as n approaches infinity. It follows from a result of Kesten [4] that $n^{3/2}P[T=n]$ has limit $\sigma/(2\pi)^{1/2}$ as n approaches infinity. In this paper we consider the asymptotic behavior of random walks conditioned by the events $[T>n]$ and $[T=n]$. Belkin [1] has obtained the result

$$\lim_{n\to\infty} P[S_n/n^{1/2} \leqq x \mid T > n] = \int_{-\infty}^{x} (|y|/2\sigma^2)\exp(-y^2/2\sigma^2)\, dy.$$

We obtain a local limit theorem which is readily seen to be a generalization of this result. Our local version is then applied to obtain the weak convergence of a sequence of probability measures on $C[0, 1]$ corresponding to a random walk conditioned by the event $[T=n]$. The limiting probability measure corresponds to a Markov process first introduced by Lévy [5] and subsequently entitled a Brownian excursion by Itô and McKean [3].

2. **A conditional local limit theorem.** Our main result is stated as

THEOREM 1. *Suppose the random variables $X_1, X_2, \cdots$ are i.i.d. on the lattice of integers with $EX_i=0$ and $EX_i^2=\sigma^2<\infty$. Then*

$$\lim_{n\to\infty}\sup_{x} |n^{1/2}P[S_n = x \mid T > n] - (|x|/2\sigma^2 n^{1/2})\exp(-x^2/2n\sigma^2)| = 0.$$

For any integer x define the hitting time $T_{\{x\}}$ either to be the first $n\geqq 1$ such that $S_n=x$ or to be $+\infty$ if no such n exists. Employing Theorem 1

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and the facts that $P[S_n=x;\ T>n]=P[T_{\{x\}}=n]$ and $n^{1/2}P[T>n]\to(2/\pi)^{1/2}\sigma$ as $n\to\infty$, we obtain

COROLLARY 1. *Under the hypotheses of Theorem* 1,

$$\lim_{n\to\infty}\sup_{x}|nP[T_{\{x\}}=n]-(|x|/\sigma n^{1/2})\phi(x/\sigma n^{1/2})|=0,$$

where $\phi(t)$ denotes the standard normal probability density function.

3. **The weak convergence of random walk conditioned by the event $[T=n]$.** On $C[0,1]$ with the uniform norm and the corresponding sigma field $\mathscr{C}$ of Borel subsets, define a sequence of probability measures $\{P_n\}$ by assigning mass

$$P[S_1/\sigma n^{1/2}=x_1,\cdots,S_n/\sigma n^{1/2}=x_n \mid T=n]$$

to the polygonal line segment ξ such that $\xi(0)=0$ and $\xi(k/n)=x_k$ for $k=0,1,\cdots,n$.

As an application of Corollary 1 we obtain

THEOREM 2. *The sequence of probability measures $\{P_n\}$ on $(C[0,1],\mathscr{C})$ converges weakly to a probability measure P which corresponds to the Brownian excursion stochastic process.*

The Brownian excursion is a Markov process with nonstationary transition density. Itô and McKean [3] discuss two alternative derivations of this process and provide explicit expressions for the transition density. Belkin [2] previously has obtained results analogous to Theorem 2 with the conditioning event $[T>n]$.

Proofs of these results will appear elsewhere.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, CHATTANOOGA, TENNESSEE 37401

Current address: 1306 Avenida Polar, Apt. B-8, Tucson, Arizona 85710