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A MEROMORPHIC FUNCTION WITH ASSIGNED NEVANLINNA DEFICIENCIES

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1. Statement of result.

THEOREM. *Let $\delta(a)$ and $\theta(a)$ be nonnegative functions defined on the extended complex plane $\hat{C}$ such that $0 \leqq \delta(a)+\theta(a) \leqq 1$, $a \in \hat{C}$,*

$$\sum_{a\in\hat{C}} \{\delta(a) + \theta(a)\} \leqq 2.$$

Then there exists a function $f(z)$ which is meromorphic in the finite z-plane with $\delta(a, f)=\delta(a)$, $\theta(a,f)=\theta(a)$, $a \in \hat{C}$. Finally, let $\phi(r)$ be a positive increasing function with

$$\phi(r) \to \infty \qquad (r \to \infty). \tag{1.1}$$

Then our function $f(z)$ may be chosen so that, in addition, its Nevanlinna characteristic satisfies

$$T(r) < r^{\phi(r)} \tag{1.2}$$

for all sufficiently large r.

Here we are using the standard notations of Nevanlinna's theory as described in [3], [6]; for example, $\theta(a,f)$ is the index of multiplicity (Verzweigungsindex) of a. Our function $f(z)$ thus provides a complete solution to the 'inverse problem' of the Nevanlinna theory (cf. [2, Chapter 7]; [9, Chapter 8]).

In general, the solution to the inverse problem must be of infinite order (cf. [8]); (1.2) asserts that this may be as 'small' an infinite order as desired.

Among earlier partial solutions to this problem we note Nevanlinna [5], Goldberg (cf. [2, Chapter 7, Theorems 8.2, 8.3]) and Fuchs-Hayman (cf. [3, §4.1]).

2. **Method of proof.** Given the function $\phi(r)$ of (1.1) and the sets $\Delta=\{a;\, \delta(a)>0\}$ and $\theta=\{a;\, \theta(a)>0\}$, we shall associate a sequence $\{r_k\}$

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with r_{k+1}/r_k tending rapidly to infinity with the property that if b is a fixed element of $\hat{C}-(\Delta\cup\theta)$ then, for all $a \in \hat{C}$,

$$|\{1-\delta(a)\}n(r,b,f)-n(r,a,f)| \leqq (4/k)n(r,b,f) \qquad (r_k \leqq r \leqq r_{k+1}), \tag{2.1}$$

$$|\theta(a)n(r,b,f)-\{n(r,a,f)-\bar{n}(r,a,f)\}| < (4/k)n(r,b,f) \qquad (r_k \leqq r \leqq r_{k+1}) \tag{2.2}$$

and

$$2^k(1-k^{-1}) \leqq n(2r,b,f)/n(r,b,f) \leqq 2^{k+1}(1+k^{-1}) \qquad (r_k \leqq r \leqq r_{k+1}). \tag{2.3}$$

That f has preassigned deficiencies and indices of multiplicity follows from (2.1), (2.2) and the fact (cf. [**6**, p. 276]) that there is a set E of logarithmic capacity zero such that

$$N(r,a,f) \sim T(r,f) \qquad (r \to \infty, a \notin E).$$

Also, if r_{k+1}/r_k increases sufficiently rapidly, (2.3) shows that the growth of $T(r,f)$ may be retarded in accord with (1.2).

One first constructs a 'quasi-meromorphic' function[2] $g(z)$ which satisfies, formally, (2.1), (2.2) and (2.3), and then factors

$$g = f \circ h \tag{2.4}$$

where f is a meromorphic function and h is a quasiconformal homeomorphism of the complex plane onto itself. The problem is to ensure that h in (2.4) sufficiently approximates the identity so that (2.1), (2.2) and (2.3) (with, perhaps, a different sequence $\{r_k\}$) transfer to f.

Using an important principle of Teichmuller [**7**], Le Van Thiem [**4**] first applied this principle to the inverse problem, and the method was further exploited by Goldberg (cf. [**2**, Chapter 7]). These efforts had two limitations: the characteristic of g had to be of finite order and the dilatation of g, $d_g(z)=|g_{\bar{z}}(z)/g_z(z)|$ was subject to

$$\iint_{|z|\geqq 1} d_g(z)\,|z|^{-2}\,dx\,dy < \infty. \tag{2.5}$$

In [**1**, Theorem 2], it was shown that this principle applies under the more flexible condition

$$D_g(r) \equiv \int_0^{2\pi} d_g(re^{i\theta})\,d\theta = o(1) \qquad (r \to \infty), \tag{2.6}$$

[2] A 'quasi-meromorphic function' is one which may be expressed as in (2.4).

and the freedom allowed by (2.6) is decisive here. For it is not hard to show that f and g will have the same deficiencies and indices of multiplicity if $D_g(r)$ decreases very rapidly with respect to $\log\{n(2r, b, g)/n(r, b, g)\}$. By increasing the ratios r_{k+1}/r_k we can diminish $D_g(r)$ with respect to $\log\{n(2r, b, g)/n(r, b, g)\}$; thus (2.5) is very unlikely to hold. Finally, g is constructed by piecing together functions discussed in [**1**] and [**5**].

ADDED IN PROOF (MARCH 15, 1974). Dr. A. A. Goldberg has informed me that the substitution of (2.6) for (2.5) first appears in the work of P. P. Belinskii. *The behavior of quasiconformal mappings at an isolated singular point*, Učen. Zap. L'vov. Gos. Univ. **29** (1954), 58–70. (Russian). However, Belinskii did not apply this to the inverse problem.

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