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A REMARK CONCERNING PERFECT SPLINES

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Let $x:=(x_i)$ be nondecreasing. For a sufficiently smooth f, denote by $f|_x:=(f_i)$ the corresponding sequence given by the rule

$$f_i := f^{(j)}(x_i) \quad \text{with } j = j(i) := \max\{m \mid x_{i-m} = x_i\}.$$

Assuming that x is in $[a, b]$ and that $x_i<x_{i+n}$, all i, $f|_x$ is defined for every f in the Sobolev space

$$W_\infty^{(n)}[a, b] := \{f \in C^{(n-1)}[a, b] \mid f^{(n-1)} \text{ abs. cont.}; f^{(n)} \in L_\infty[a, b]\}.$$

Karlin [6] discusses the problem of minimizing $\|f^{(n)}\|_\infty$ over all f in $\Pi(x, \alpha):=\{f \in W_\infty^{(n)} | f|_x=\alpha\}$ for a given sequence α, and announces the following

THEOREM (S. KARLIN [6]). *Let $x=(x_i)_1^{n+r}$ be a given nondecreasing sequence in the finite interval $[a, b]$, with $x_i<x_{i+n}$, all i. Let $\alpha \in R^{n+r}$ be given. Then $\Pi(x, \alpha)$ contains a perfect spline of order n with less than r (interior) knots, i.e., a function of the form*

$$(1) \qquad p(x) = \sum_{i=0}^{n-1} a_i x^i + c\left[x^n + 2\sum_{i=1}^{k-1} (-)^i (x - \xi_i)_+^n\right]$$

for some real constants $a_0, \cdots, a_{n-1}$, and c, and for $a<\xi_1<\cdots<\xi_{k-1}<b$ with $k \leqq r$. Further, $\|f^{(n)}\|_\infty$ takes on its minimum value over $f \in \Pi(x, \alpha)$ at this p.

It is the purpose of this note to outline a simple proof of this theorem.

For this, denote by $[x_i, \cdots, x_{i+n}]f$ the nth divided difference of f at the $n+1$ points $x_i, \cdots, x_{i+n}$. Then $[x_i, \cdots, x_{i+n}](f-g)=0$ for all f, $g \in \Pi(x, \alpha)$ and $i=1, \cdots, r$. Further, it is well known (see e.g., [2]) that, for $f \in W_1^{(n)}[a, b]$,

$$[x_i, \cdots, x_{i+n}]f = \int_a^b \varphi_i(t) f^{(n)}(t)\, dt$$

with

$$\varphi_i(t) := M_{i,n}(t)/n! := [x_i, \cdots, x_{i+n}](\cdot - t)_+^{n-1}/(n-1)!$$

a (polynomial) B-spline of order n having the knots $x_i, \cdots, x_{i+n}$. Hence,

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with f_0 any particular element of $\Pi(x, \alpha)$, $\Pi(x, \alpha)$ is contained in

$$(2) \qquad \left\{ f \in W_\infty^{(n)} \,\middle|\, \int_a^b \varphi_i(t)(f - f_0)^{(n)}(t)\, dt = 0,\ i = 1, \cdots, r \right\}.$$

On the other hand, for every f in the set (2), there exists a polynomial p_f of degree $<n$ so that $f - p_f \in \Pi(x, \alpha)$, viz. the unique polynomial p_f of degree $<n$ for which

$$p_f|_{(x_i)_1^n} = (f - f_0)|_{(x_i)_1^n}.$$

Consequently, with the definition

$$\Pi^{(n)}(x, \alpha) := \left\{ g \in L_\infty[a, b] \,\middle|\, \int \varphi_i g = \int \varphi_i f_0^{(n)},\ i = 1, \cdots, r \right\},$$

it follows that

$$(3) \qquad \inf\{\|f^{(n)}\|_\infty \mid f \in \Pi(x, \alpha)\} = \inf\{\|g\|_\infty \mid g \in \Pi^{(n)}(x, \alpha)\},$$

and that n-fold differentiation maps the set of solutions of the left-hand minimization problem one-one and onto the set of solutions of the right-hand minimization problem. Equation (3) can already be found in Favard's pioneering paper [3].

It remains to show that $\Pi^{(n)}(x, \alpha)$ contains a function of constant absolute value and with less than r sign changes, and which solves the right-hand minimization problem in (3). For this, we use the idea, apparently due to M. G. Krein [7], of looking at constrained minimization dually, as a problem of finding norm preserving extensions for a given linear functional, and then using representation theorems for such functionals. Consider the linear functional λ_0 defined on

$$S_{n,x} := \operatorname{span}(\varphi_1, \cdots, \varphi_r) \subseteq L_1[a, b]$$

by the rule

$$\lambda_0 \varphi := \int_a^b \varphi(t) f_0^{(n)}(t)\, dt, \quad \text{all } \varphi \in S_{n,x}.$$

Then, identifying $L_\infty[a, b]$ with the continuous dual of $L_1[a, b]$ in the usual way, $\Pi^{(n)}(x, \alpha)$ is seen to coincide with the collection of all extensions of λ_0 to a continuous linear functional on $L_1[a, b]$. Hence

$$\inf\{\|g\|_\infty \mid g \in \Pi^{(n)}(x, \alpha)\} = \inf\{\|\lambda\| \mid \lambda \in (L_1[a, b])^*,\ \lambda|_{S_{n,x}} = \lambda_0\} = \|\lambda_0\|$$

by the Hahn-Banach theorem, settling existence of a solution as well. Let ψ be an element of $S_{n,x}$ of unit 1-norm at which λ_0 takes on its norm, i.e., let

$$\psi \in S_{n,x}, \qquad \|\psi\|_1 = 1, \qquad \lambda_0 \psi = \sup_{\varphi \in S_{n,x}} \lambda_0 \varphi / \|\varphi\|_1,$$

and set $h := (\lambda_0\psi)\text{signum}\,\psi$. If g is any point in $\Pi^{(n)}(\boldsymbol{x}, \boldsymbol{\alpha})$ at which the infimum is taken on, then

$$\|g\|_\infty = \|\lambda_0\| = \lambda_0\psi = \int \psi g \leqq \|\psi\|_1 \|g\|_\infty = \|g\|_\infty$$

and equality must therefore hold in Hölder's inequality. This implies that

$$g(t) = \|g\|_\infty \text{ signum } \psi(t) = h(t) \quad \text{a.e. on } \{t \mid \psi(t) \neq 0\}.$$

Hence, *if ψ vanishes only on a set of measure zero, then h is an element of $\Pi^{(n)}(\boldsymbol{x}, \boldsymbol{\alpha})$ of constant absolute value and is the unique element of $\Pi^{(n)}(\boldsymbol{x}, \boldsymbol{\alpha})$ at which* $\inf\{\|g\|_\infty | g \in \Pi^{(n)}(\boldsymbol{x}, \boldsymbol{\alpha})\}$ *is attained.*

This simple idea is at the bottom of Glaeser's successful treatment [**4**] of the special case

$$r = n, \qquad x_1 = \cdots = x_n = a, \qquad x_{n+1} = \cdots = x_{2n} = b.$$

In this special case, $S_{n,\boldsymbol{x}}$ reduces to the space of polynomials of degree $<r$, hence every nonzero $\psi \in S_{n,\boldsymbol{x}}$ vanishes only at $<r$ points.

In the general case, $\psi \in S_{n,\boldsymbol{x}}$ is known to vanish only at $<r$ points unless ψ vanishes on an interval. But whether or not this happens, with $S_\varepsilon := K_\varepsilon(S_{n,\boldsymbol{x}}) \subseteq L_1[a, b]$, where $\varepsilon > 0$ and

$$(K_\varepsilon g)(x) := \int_{-\infty}^{\infty} \exp(-(x - \xi)^2/(2\varepsilon^2))g(\xi)\, d\xi/(\varepsilon\sqrt{2\pi}),$$

every nonzero $\psi \in S_\varepsilon$ vanishes only at $<r$ points [**5**, proof of Theorem 4.1 in Chapter 10, especially item (4.23)]. Hence, there exists exactly one h_ε in

$$\Pi_\varepsilon := \left\{ g \in L_\infty[a, b] \,\middle|\, \int \varphi_\varepsilon g = \int \varphi_\varepsilon f_0^{(n)}, \text{ all } \varphi_\varepsilon \in S_\varepsilon \right\}$$

at which $\inf\{\|g\|_\infty | g \in \Pi_\varepsilon\}$ is attained, and this h_ε is of constant absolute value and has fewer than r sign changes. Since

$$\lim_{\varepsilon \to 0+} \|\varphi - K_\varepsilon \varphi\|_1 = 0 \quad \text{for all } \varphi \in S_{n,\boldsymbol{x}},$$

it follows that

$$\liminf_{\varepsilon \to 0+} \|h_\varepsilon\|_\infty \leqq \inf\{\|g\|_\infty \mid g \in \Pi^{(n)}(\boldsymbol{x}, \boldsymbol{\alpha})\},$$

hence, for some positive null sequence (ε_m) and some points $\xi_1 < \cdots < \xi_{k-1}$ in $[a, b]$ with $k \leqq r$, (h_{ε_m}) converges uniformly on compact subsets of $[a, b] \backslash \{\xi_1, \cdots, \xi_{k-1}\}$ to some function h for which

$$\lim_{m \to \infty} \|h_{\varepsilon_m}\|_\infty = \|h\|_\infty \leqq \inf\{\|g\|_\infty \mid g \in \Pi^{(n)}(\boldsymbol{x}, \boldsymbol{\alpha})\}.$$

But this h is necessarily of constant absolute value, has fewer than r sign changes and is in $\Pi^{(n)}(x, \alpha)$, which finishes the proof.

The above argument extends at once to the minimization of $\|Lf\|_\infty$ under the same constraints, with L an nth order ordinary linear differential operator which is totally disconjugate.

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