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ZEROS OF FUNCTIONS IN THE BERGMAN SPACES

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A function $f(z)$ analytic in the unit disk is said to belong to the Bergman space A^p $(0<p<\infty)$ if $\int_0^1\int_0^{2\pi}|f(re^{i\theta})|^p r\,dr\,d\theta<\infty$. It is clear that A^p contains the Hardy space H^p of analytic functions for which $\lim_{r\to 1}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta<\infty$. We adopt the convention that $A^\infty=H^\infty$, the space of bounded analytic functions in the disc.

Assuming that $f(0)\neq 0$, we list the zeros of f in order of nondecreasing modulus: $0<|z_1|\leqq|z_2|\leqq|z_3|\leqq\cdots<1$. We repeat z_i according to the multiplicity of the zero of f at z_i. The sequence $\{z_i\}$ is called the zero set of f. If $f\in A^p$ (resp. H^p), then z_i will be called an A^p (resp. H^p) zero set. It has long been known that H^p zero sets $(0<p\leqq\infty)$ are completely characterized by the condition $\prod_{k=1}^\infty 1/|z_k|<\infty$. (Equivalently, $\sum_{k=1}^\infty 1-|z_k|<\infty$.) In particular, the condition is independent of p. Our results show that the situation for A^p zero sets is considerably more complex.

LEMMA 1. *If $\{z_k\}$ is an A^p zero set $(0<p<\infty)$, then*

$$\prod_{K=1}^{N}\frac{1}{|z_k|}=O(N^{1/p}).$$

COROLLARY. *If $\{z_k\}$ is an A^p zero set $(0<p<\infty)$, then for each $\varepsilon>0$,*

$$\sum_{k=1}^{\infty}(1-|z_k|)\left\{\log\frac{1}{1-|z_k|}\right\}^{-1-\varepsilon}<\infty.$$

If $f(z)=\sum_{n=0}^\infty a_n z^n$, let $S_N^{(p)}=\sum_{k=1}^N |a_k|^p$, $p>0$.

LEMMA 2. *If $S_N^{(2)}=O(N^\alpha)$ for some $\alpha\geqq 1$, then $f\in A^p$ for all $p<2/\alpha$.*

LEMMA 3. *For some p, $1\leqq p\leqq 2$, suppose that $\sum_{N=1}^\infty N^{-p}S_N^{(p)}<\infty$ and $N^{1-p}S_N^{(p)}=O(1)$. Then $f\in A^{p'}$, $1/p+1/p'=1$.*

Lemma 1 is proved by an application of Jensen's theorem. Lemmas 2 and 3 follow from corresponding coefficient conditions, after a summation by parts. In particular, Lemma 3 is a consequence of the fact that

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$\sum |a_N/N| N < \infty$ implies $f \in A^\infty$, and that $\sum |a_N/N|^2 N < \infty$ implies $f \in A^2$. One merely applies the Riesz interpolation theorem and summation by parts to obtain the result.

THEOREM 1. *Let $0<p<q\leqq\infty$. Then there exists an A^p zero set which is not an A^q zero set.*

SKETCH OF PROOF. Let $f(z)=\prod_{k=0}^{\infty} 1+uz^{b^k}$, where b is an integer greater than 2, and u is a positive constant. Using the notation of the lemmas, one verifies that:

(1) Every partial product for $f(z)$ is a partial sum of its Taylor series.

(2) If $N=\sum_{k=0}^{s-1} b^k$, $S_N^{(p)}=(1+u^p)^s$.

(3) If $u>1$, if $N=\sum_{k=0}^{s-1} b^k$, and if $\{z_i\}$ are the ordered zeros of f, then $\prod_{i=1}^{N} 1/|z_i|=u^s$.

From these facts, and from Lemmas 1, 2 and 3, we conclude that:

(4) If $b\leqq 1+u^2$, then $f\in A^p$ for all $p<2\log b/\log(1+u^2)$. (Also, in this case, $f\notin A^2$.)

(5) If $1+u^s\leqq b^{s-1}$ for some s, $1<s\leqq 2$, $f\in A^p$ for all $p<s'$, where $1/s+1/s'=1$.

(6) If $u>1$, the zero set of f is not the zero set of any function in A^q for $q>\log b/\log u$.

An examination of (4), (5) and (6) shows that if $0<p<q\leqq\infty$, u and b may always be chosen to yield a function f in A^p whose zero set is not an A^q zero set.

THEOREM 2. *For $0<p<\infty$, the union of two A^p zero sets is not in general an A^p zero set.*

To prove Theorem 2, we choose one of the functions $f\in A^p$ constructed in Theorem 1, with the parameter $u>1$. We choose a positive integer N and require that each zero of f be repeated N times. For N sufficiently large we obtain a sequence which, by Lemma 1, cannot be an A^p zero set.

We state two corollaries to the above theorems, both of which again contrast sharply with H^p theory.

COROLLARY (TO THEOREM 1). *It is not possible to represent an arbitrary A^1 function as the product of two functions in A^2, one of them nonvanishing.*

COROLLARY (TO THEOREM 2). *Consider the operator M_z of multiplication by z on A^2 (a weighted unilateral shift). There exist two nontrivial closed invariant subspaces of M_z whose intersection is trivial.*

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