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THE MINIMUM NUMBER OF EDGES AND VERTICES IN A GRAPH WITH EDGE CONNECTIVITY n AND m n-BONDS

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The purpose of this note is to announce the closed form solution of the following extremal problem in graph connectivity (see [1] and [2]): compute the minimum number of edges and vertices in a graph with a given edge connectivity and a given number of minimum cardinality bonds. It is hoped that the solution to this problem will be helpful in work on the following apparently quite difficult problem posed by Van Slyke and Frank [7]: given the number of k-sets of edges of a graph containing bonds, what are bounds for the number of k'-sets of edges containing bonds, $k' \neq k$?

G denotes a graph with vertex and edge sets $V(G)$ and $E(G)$, both finite. Loops and multiple edges are allowed. A polygon of G is a connected subgraph of valency two. A multigon is a subgraph with at least two edges, that is either a polygon or a link graph after identification of multiple edges. A bond is a minimal nonempty set of edges that meets no polygon in just one edge. We define the edge connectivity $\lambda(G)$ of G to be 0 if G is disconnected, null or $|V(G)|=1$. Otherwise

$$\lambda(G) = \min\{|B| \mid B \text{ is a bond of } G\}.$$

For n and m positive integers put

$$C(G) = |\{B \subseteq E(G) \mid B \text{ is a } \lambda(G)\text{-bond}\}|,$$

$$E(n, m) = \min\{|E(G)| \mid \lambda(G) = n \text{ and } C(G) = m\},$$

$$V(n, m) = \min\{|V(G)| \mid \lambda(G) = n \text{ and } C(G) = m\}.$$

For definitions of terms not defined here the reader is referred to Tutte [5], [6].

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THEOREM A. *For n and m positive integers, and all r_i integer*

(a1) $E(1, m) = m$,

(a2) $V(1, m) = m + 1$;

(b1)
$$E(2, m) = q_1(m) = \min\left\{r_0 + r_1 + \cdots + r_k \,\middle|\, \binom{r_0}{2} + \binom{r_1}{2} + \cdots + \binom{r_k}{2} = m, \min r_i \geqq 2\right\},$$

(b2)
$$V(2, m) = q_2(m) = \min\left\{r_0 + r_1 + \cdots + r_k - k \,\middle|\, \binom{r_0}{2} + \binom{r_1}{2} + \cdots + \binom{r_k}{2} = m, \min r_i \geqq 2\right\};$$

(c1)
$$E(n, m) = \min\left\{n\left(1 + \frac{n+1}{2n} r_2 + r_4 + 2r_6\right) \,\middle|\, 1 + r_2 + 3r_4 + 5r_6 = m, \min r_i \geqq 0\right\},$$

(c2)
$$V(n, m) = \min\{2 + r_2 + 2r_4 \mid 1 + r_2 + 3r_4 = m, \min r_i \geqq 0\} \quad \textit{for odd } n \geqq 3;$$

(d1)
$$E(n, m) = \min\left\{\frac{n}{2}(r_0 + r_1 + \cdots + r_k + \delta) + \delta' \,\middle|\, \binom{r_0}{2} + \binom{r_1+2}{2} + \binom{r_2+2}{2} + \cdots + \binom{r_k+2}{2} - k + \delta = m\right\},$$

(d2)
$$V(n, m) = q^*(m) = \min\left\{r_0 + r_1 + \cdots + r_k + \delta \,\middle|\, \binom{r_0}{2} + \binom{r_1+2}{2} + \binom{r_2+2}{2} + \cdots + \binom{r_k+2}{2} - k + \delta = m\right\}$$

for even $n \geqq 4$

where $k \geqq 0$, $\min\{r_0-2, r_1, \cdots, r_k\} \geqq 0$, $\delta = 0$ *or* 1 *and* $\delta' = \delta$ *except when* $n \geqq 6$, *and* (i) $r_k = 1$ *or* (ii) $k = 0$ *and* $r_0 = 3$, *in which case* $\delta' = 0$.

Part (a) of the theorem is trivial. A simple proof of (b) can be given by defining the following equivalence relation: for $e_1, e_2 \in E(G)$ we say $e_1 \sim e_2$ if either $\{e_1, e_2\}$ is a 2-bond or intersects no 2-bond. In fact, the proof given by this equivalence relation implies (b1) for general matroids. However, the proof cannot be generalized to higher connectivities since

Theorem A(c1) and (d1) are false even for regular (unimodular) matroids. This can be seen by considering the Kuratowski graphs K_5 and $K_{3,3}$.

Parts (c) and (d) are the heart of Theorem A. For a proof of (c) and (d) we first construct the appropriate graphs to show that the right-hand sides provide upper bounds. To complete the proof we use the notion of crossing bonds. Given G and bonds B_1 and B_2, we say B_1 crosses B_2 if B_1 intersects two components of the graph produced from G by removing B_2. The main results on crossing bonds are Lemmas 1 and 2 below. These lemmas allow us to apply induction in (c) and (d).

LEMMA 1. *If $\lambda(G)$ is odd then G has no crossing $\lambda(G)$-bonds.*

LEMMA 2. *Suppose $\lambda(G)\geqq 2$ is even. If G has a $\lambda(G)$-bond that is not the star of a vertex, then either G is a multigon with adjacent vertices joined by $\lambda(G)/2$ edges, or G has a $\lambda(G)$-bond that is not crossed by a $\lambda(G)$-bond and is not the star of a vertex.*

The notion of a $\lambda(G)$-bond which is not crossed by a $\lambda(G)$-bond resembles somewhat that of a hinge [**5**, p. 118]. In particular we note the similarity of Lemma 2 and [**5**, Theorem 11.34].

Using Theorem A it is not hard to prove that for each pair of positive integers $n\neq 2$ and m there is a G with $\lambda(G)=n$, $C(G)=m$, $|E(G)|=E(n, m)$ and $|V(G)|=V(n, m)$. However, for $n=2$ this result is false. The smallest example corresponds to $m=83$. In fact, the unique G having $\lambda(G)=2$, $C(G)=83$ and $|E(G)|=E(2, 83)=19$ has $|V(G)|=18>V(2, 83)=13+3+2+2-3=17$.

Theorem A has the drawback that in (b)–(d) we only implicitly compute $E(n, m)$ and $V(n, m)$. Theorem B is an attempt to remedy this shortcoming.

Let m be a positive integer. We say that a sequence of integers $(r_0, r_1, \cdots, r_k)$ with $\min r_i \geqq 2$ is a 2-decomposition of m if

$$\binom{r_0}{2}+\binom{r_1}{2}+\cdots+\binom{r_k}{2}=m.$$

A 2-decomposition is maximal if r_0 is taken as large as possible, then r_1 is taken as large as possible, etc. A sequence of integers $(r_0, r_1, \cdots, r_k, \delta)$ with $\min\{r_0-2, r_1, \cdots, r_k\}\geqq 0$ and $\delta=0$ or 1 is a 2*-decomposition of m if

$$\binom{r_0}{2}+\binom{r_1+2}{2}+\binom{r_2+2}{2}+\cdots+\binom{r_k+2}{2}-k+\delta=m.$$

Maximal 2*-decomposition is defined analogously to maximal 2-decomposition. Maximal decompositions are similar to the "r-canonical representations" used by Katona [3] and Kruskal [4]. Let $(s_0, s_1, \cdots, s_k)$ be a maximal 2-decomposition of m, and let $(s_0^*, s_1^*, \cdots, s_l^*, \delta)$ be a maximal 2*-decomposition of m. Put

$$p_1(m) = s_0 + s_1 + \cdots + s_k, \qquad p_2(m) = s_0 + s_1 + \cdots + s_k - k,$$
$$p^*(m) = s_0^* + s_1^* + \cdots + s_l^* + \delta.$$

THEOREM B. *For positive integers n, m and k*
(a1) $E(1, m) = m$,
(a2) $V(1, m) = m+1$;
(b1) $p_1(m) - 4 \leqq E(2, m) \leqq p_1(m)$,
(b2) $p_2(m) - 1 \leqq V(2, m) \leqq p_2(m)$;
(c1) $E(n, 3k+l) = [n(2k+l+1)/2 + \frac{1}{2}]$,
(c2) $V(n, 3(k-1)+l) = 2(k-1)+l+1$ *where* $n \geqq 3$ *is odd and* $l = 1, 2, 3$;
(d1) $np^*(m)/2 \leqq E(n, m) \leqq np^*(m)/2 + 1$,
(d2) $V(n, m) = p^*(m)$ *where* $n \geqq 4$ *is even.*

In Theorem B(c1), $[\cdot]$ is the greatest integer function (e.g., $[3/2]=1$). The formula given in (c1) is false for $k=0$ and $l=3$ since $E(n, 3)=2n+1$ for odd $n \geqq 3$.

Theorem B(a) is identical to Theorem A(a). Part (c) of Theorem B is easy to prove. The minimization problems in Theorem A(c) are in fact knapsack problems. To prove (b) and (d) it suffices by Theorem A(b) and (d) to prove $p_1(m) - 4 \leqq q_1(m)$, $p_2(m) - 1 \leqq q_2(m)$ and $p^*(m) \leqq q^*(m)$ (i.e., $p^*(m) = q^*(m)$). These bounds are best possible. (E.g., $p_1(594) = 34+8+3+2+2 = 49$, but $q_1(594) = 33+12 = 45$. $m = 594$ is the smallest such example.)

We indicate the proof of $p_1(m) - 4 \leqq q_1(m)$. For $n \geqq 2$ an integer we define $m_1(n) = \min\{m \mid p_1(m) = n\}$. Clearly $m_1(n) \leqq \binom{n}{2}$. It can be shown that every integer $b \geqq 4$ is uniquely representable in the form $b = n+k$ where $n \geqq 2$ and k are integers such that

$$\begin{aligned} m_1(n) + 1 \leqq k &\leqq m_1(n+1) + 1 \quad \text{if } m_1(n+1) < m_1(n+2), \quad \text{and} \\ &\leqq m_1(n+2) + 2 \quad \text{if } m_1(n+2) < m_1(n+1). \end{aligned}$$

Further $m_1(n+k) = \binom{k}{2} + m_1(n)$.

One can prove using this recursion that $m_1(n+4) > \frac{1}{2}\binom{n}{2}$ for all $n \geqq 2$. But if $p_1(m) - 4 > q_1(m)$ for some m, then by considering the first such m we can find an n for which $m_1(n+4) \leqq \frac{1}{2}\binom{n}{2}$. This contradiction establishes $p_1(m) - 4 \leqq q_1(m)$.

One can dualize Theorem B for planar graphs. The resulting theorem computes the minimum number of edges and vertices in a planar graph having given (polygon) girth and a given number of waist circuits.

Theorem A can be used in a straightforward way to compute the maximum number of minimum cardinality bonds in a graph with a given number of edges or vertices and given edge connectivity.

Complete proofs will appear elsewhere.

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