BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 4, July 1974

A REMARK ON THE LINDELÖF HYPOTHESIS

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Communicated by F. W. Gehring, November 12, 1973

I. **Introduction.** In this note, we sketch a development which offers new insight into some previous work on the Lindelöf hypothesis (LH).

As is well known [**5**, p. 276], the following two statements are equivalent to the LH:

$$\int_0^T |\zeta(\tfrac{1}{2} + it)|^{2k}\, dt = O(T^{1+\varepsilon}) \quad \text{for each } \varepsilon > 0 \text{ and } k \geqq 1; \tag{1}$$

$$\int_0^\infty |\zeta(\tfrac{1}{2} + it)|^{2k} e^{-\delta t}\, dt = O(\delta^{-1-\varepsilon}) \quad \text{for each } \varepsilon > 0 \text{ and } k \geqq 1. \tag{2}$$

That (1)$\Leftrightarrow$(2) follows from an elementary Tauberian argument. At present, (1) and (2) are known only for $k=1$, $k=2$.

According to the general formalism of Titchmarsh [**5**, pp. 137–138],

$$\int_0^\infty |\zeta(\tfrac{1}{2} + it)|^{2k} e^{-2\delta t}\, dt = O(1) + \int_0^\infty |\phi_k(ixe^{-i\delta})|^2\, dx,$$

where $\phi_k(z) = \sum_{n=1}^\infty d_k(n)e^{-nz}$+residue term. Hopefully, one could expand the ϕ_k integral and estimate the resulting infinite series. This does not seem feasible, however, unless $\phi_k(z)$ satisfies a certain approximate functional equation (AFE); see [**5**, p. 147] and [**6**, p. 42]. This is one reason why only $k=1$, $k=2$ are known.

However, Bellman [**2**] has shown that, if the e^{-nz} in $\phi_k(z)$ are replaced by so-called Voronoi functions, one will always get an AFE. Unfortunately, these Voronoi functions have proved too messy to be useful computationally.

It would therefore be of interest to see what could be done with a method which involves much simpler functions.

II. **Development of the main theorem.** We base our development on the series

$$\sum_{n=1}^{\infty} n^{Q/A} d_k(n) \exp(-zn^{1/A})$$

AMS (MOS) subject classifications (1970). Primary 10H05.

[1] This work was done while at Stanford University. Preparation of the manuscript was supported in part by the U.S. Army Research Office (Durham).

for appropriate values of Q and A, $A>0$, $A+2Q-1\geqq 0$. The basic identity is:

$$(3)\qquad \begin{aligned}\phi_k(z) &= \frac{A}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\Gamma(As+Q)\zeta^k(s)z^{-As}\,ds\\ &= \sum_{n=1}^{\infty}(zn^{1/A})^Q\, d_k(n)\exp(-zn^{1/A}) + z^{-A}P_{k-1}(\log z),\end{aligned}$$

where $\mathrm{Re}(z)>0$ and $P_m(u)$ denotes some polynomial of degree m. Following Titchmarsh, we let $z=ixe^{-i\delta}$, $0<\delta\leqq\pi/2$, $0<x<\infty$, and study $\delta\to 0$. Using Parseval's formula for Mellin transforms and the Stirling approximation for $\Gamma(w)$, we readily check that

$$(4)\qquad \begin{aligned}&\int_0^\infty |\phi_k(ixe^{-i\delta})|^2\, x^{A-1}\,dx\\ &\quad = O(1) + \int_0^\infty \left|\zeta\left(\frac{1}{2}+i\frac{t}{A}\right)\right|^{2k} e^{-2\delta t}t^{A+2Q-1}\left[1+O\left(\frac{1}{1+t}\right)\right]dt.\end{aligned}$$

It is important to determine an appropriate AFE for $\phi_k(z)$. This is where an optimal choice of A is essential. An uninspired calculation using the FE for $\zeta(s)$ and the stirling approximation shows that when

$$(5)\qquad A = k/2$$

we have the L_2 AFE,

$$(6)\qquad \phi_k(1/z) = \mu_k z^A[\phi_k(4\pi^2A^2z) + e_k(4\pi^2A^2z)],$$

where

$$(7)\qquad e_k(z) = \frac{A}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\Gamma(As+Q)\zeta^k(s)R_k(s)z^{-As}\,ds$$

and $R_k(s)=O(s^{-1})$, $\mu_k=\exp[-i\pi Q+\frac{1}{2}\pi i(1-k)+\frac{1}{2}k\ln(k\pi)]$.[2] It follows that

$$(8)\qquad \int_0^\infty |\phi_k(ixe^{-i\delta})|^2\, x^{A-1}\,dx = \int_{2\pi A}^\infty |\phi_k|^2\, x^{A-1}\,dx + \int_{2\pi A}^\infty |\phi_k+e_k|^2\, x^{A-1}\,dx.$$

We contend that e_k is L_2 negligible in (8), so that (6) is indeed an L_2 AFE. To see this, observe that by Parseval's formula again,

$$\begin{aligned}\int_0^\infty |e_k(ixe^{-i\delta})|^2\, x^{A-1}\,dx = O(1) &+ \int_0^\infty \left|R_k\left(\frac{1}{2}+i\frac{t}{A}\right)\right|^2\left|\zeta\left(\frac{1}{2}+i\frac{t}{A}\right)\right|^{2k}\\ &\cdot e^{-2\delta t}t^{A+2Q-1}\left[1+O\left(\frac{1}{1+t}\right)\right]dt.\end{aligned}$$

[2] For $k=1$ and $Q=0$, (6) essentially reduces to a theta identity.

Since $R_k(s)=O(s^{-1})$,

$$\int_{2\pi A}^{\infty} |e_k|^2 x^{A-1}\,dx \leqq \int_0^{\infty} |e_k|^2 x^{A-1}\,dx = o[W(\delta)], \tag{9}$$

where

$$W(\delta) = \int_0^{\infty} t^{A+2Q-1}\left|\zeta\left(\frac{1}{2}+i\frac{t}{A}\right)\right|^{2k} e^{-2\delta t}\,dt. \tag{10}$$

We note here that, since $A+2Q-1\geqq 0$, $\lim_{\delta\to 0} W(\delta)=\infty$ [5, p. 151]. By means of (4), (8), (9) and simple L_2 estimates, we check that

$$W(\delta) \sim \int_0^{\infty} |\phi_k(ixe^{-i\delta})|^2 x^{A-1}\,dx \sim 2\int_{2\pi A}^{\infty} |\phi_k|^2 x^{A-1}\,dx. \tag{11}$$

Thus, e_k is indeed L_2 negligible.

At the same time, a simple calculation shows that the residue term $z^{-A}P_{k-1}(\log z)$ also drops out:

$$W(\delta) \sim 2\int_{2\pi A}^{\infty}\left|\sum_{n=1}^{\infty} n^{Q/A} d_k(n)\exp(-zn^{1/A})\right|^2 x^{A+2Q-1}\,dx. \tag{12}$$

The obvious hope is that we can expand the right side of (12) and then estimate the resulting infinite series. It is immediately seen that the terms of the series involve integrals of the form

$$\int_{2\pi A}^{\infty} e^{-ax}\cos bx \cdot x^{A+2Q-1}\,dx.$$

This integral has its simplest value when $A+2Q-1=0$. For this reason, we shall now assume that

$$A + 2Q - 1 = 0. \tag{13}$$

We will thus arrive at the following result.

THEOREM. *The LH is valid if and only if*

$$\int_{2\pi A}^{\infty}\left|\sum_{n=1}^{\infty} n^{Q/A} d_k(n)\exp(-zn^{1/A})\right|^2 dx = O(\delta^{-1-\varepsilon})$$

for each $\varepsilon>0$ *and integer* $k\geqq 1$. *Here* $Q=\frac{1}{2}(1-A)$, $A=\frac{1}{2}k$, *and* $z=ixe^{-i\delta}$.

Incidentally, the same result holds when only even values of k are considered, as is apparent by going back to (1) and (2).

III. **Computations.** Let us now expand the integral in the Theorem.

We obtain:

$$\int_{2\pi A}^{\infty}\left|\sum_{n=1}^{\infty} n^{Q/A}\, d_k(n)\exp(-zn^{1/A})\right|^2 dx$$

$$\begin{aligned}(14)\qquad &= \sum_{n=1}^{\infty} D_k(n)^2\int_{2\pi A}^{\infty}\exp(-2n^{1/A}x\sin\delta)\,dx\\ &\quad + 2\sum_{m=2}^{\infty}\sum_{n=1}^{m-1} D_k(m)D_k(n)\int_{2\pi A}^{\infty}[\cdots]\,dx\\ &= (\mathrm{I}) + 2(\mathrm{II}),\end{aligned}$$

where

$$D_k(n) = n^{Q/A}\, d_k(n) = d_k(n)/n^{1/2-1/2A}$$

and

$$[\cdots] = \exp[-x(m^{1/A} + n^{1/A})\sin\delta]\cdot\cos[x(m^{1/A} - n^{1/A})\cos\delta].$$

By using $d_k(n)=O(n^{\varepsilon})$ and some simple estimates, one quickly proves that

$$(15)\qquad (\mathrm{I}) = O(\delta^{-1-\varepsilon}).$$

We now study (II) for $k\geqq 3$. See [5, pp. 143–145] for the case $k=2$. Recall that

$$\int_c^{\infty} e^{-ax}\cos bx\,dx = \frac{e^{-ac}(a\cos bc - b\sin bc)}{a^2+b^2},\qquad a>0.$$

Therefore,

$$(16)\qquad \left|\int_{2\pi A}^{\infty}[\cdots]\,dx\right| \leqq \frac{\exp[-2\pi A(m^{1/A}+n^{1/A})\sin\delta]}{[(m^{1/A}+n^{1/A})^2\sin^2\delta + (m^{1/A}-n^{1/A})^2\cos^2\delta]^{1/2}}.$$

By use of this estimate, $d_k(n)=O(n^{\varepsilon})$, and some careful computations, it is possible to prove that

$$(17)\qquad \sum_{m^{1/A}\delta\geqq P}\sum_{n=1}^{m-1} D_k(m)D_k(n)\left|\int_{2\pi A}^{\infty}[\cdots]\,dx\right| = O(\delta^{-1-\varepsilon}),$$

where $P=P(\delta)\approx(A-1)\ln 1/\delta$.

We now run into a serious difficulty. Let $R(m, n, \delta)$ denote the right side of (16). An easy computation shows that

$$\begin{aligned}(18)\qquad &\sum_{m^{1/A}\delta\leqq 1}\sum_{n=1}^{m-1} D_k(m)D_k(n)R(m,n,\delta)\\ &\qquad \geqq \sum_{m^{1/A}\delta\leqq 1}\sum_{n=1}^{m-1} m^{Q/A}n^{Q/A}R(m,n,\delta) \geqq c_A\delta^{-A},\end{aligned}$$

where $c_A > 0$ is a constant dependent only on A. This implies that estimates of a more refined (sign-dependent) nature are necessary for the further study of (II).

IV. **Concluding remarks.** The estimates needed to complete the study of (II) seem to be very difficult. One reason for this is that the terms $m^{1/A}$, $n^{1/A}$ do not transform very well under addition. This problem, however, is minor compared to the one caused by the irregular behavior of the arithmetic function $d_k(n)$. That is, the main difficulty is actually number-theoretic in nature. Compare [**4**, p. 297].

We also observe that difficulties of a similar nature are encountered when one attempts to estimate the explicit formulas which arise in the divisor problem (see [**1**], [**3**], [**7**]). Recall here the well-known equivalence relating the LH to the divisor problem [**5**, p. 278].

It therefore seems unlikely that the LH could ever be proved by estimating (II). This situation would change, however, if some way were found to effectively smooth out the $d_k(n)$. Several preliminary attempts have failed.

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