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AUTOMORPHIC MAPPINGS IN R^n

BY O. MARTIO AND U. SREBRO

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1. By an automorphic mapping in R^n we mean a continuous, open, discrete, and sense-preserving mapping f from a domain D in R^n into $\bar{R}^n = R^n \cup \{\infty\}$ which satisfies $f \circ g = f$ for all $g \in G$ for some discrete group G of n-dimensional Möbius transformations, $n \geqq 2$. The results presented here indicate differences (see §5) as well as similarities (see §4) between automorphic functions in C and automorphic mappings of bounded dilatation in R^n, $n > 2$. By mappings of bounded dilatation we mean *quasimeromorphic* (qm) mappings (cf. [MRV 1–2]).

2. Let G be a discrete Möbius group acting on the unit ball B^n. For $x_0 \in B^n$ which is not fixed by any element of $G \backslash \{\text{id}\}$ the set $P = \{x \in B^n : d(x, x_0) < d(x, g(x_0)), \forall g \in G \backslash \{\text{id}\}\}$ is a *normal fundamental polyhedron*; d denotes the hyperbolic distance. If the hyperbolic measure $V(B^n/G)$ of B^n/G is finite, then every normal fundamental polyhedron P has a finite number of $(n-1)$-faces and a finite number of *boundary vertices* $\{p_1, \cdots, p_k\} = \bar{P} \cap \partial B^n$ [S]. The last set is void when B^n/G is compact. P is said to be *simple* if for every boundary vertex $p \in \bar{P} \cap \partial B^n$ all the $(n-1)$-faces of P which meet at p are pairwise G-equivalent. By a recent result of Leon Greenberg (unpublished) it can be shown [MS] that if $V(B^n/G) < \infty$, then every point $b \in \partial B^n$ which is fixed by a parabolic element $g \in G$ is a boundary vertex of some simple fundamental polyhedron. A Möbius transformation is called *parabolic* if it has a unique fixed point in $\bar{R}^n$.

Complete proofs of the following theorems and related results will appear in [MS].

3. The existence of automorphic meromorphic functions for Möbius groups in C is usually proved by methods which cannot be used in R^n, $n > 2$. However, with a suitable modification of a construction by J. W. Alexander [A] we obtain

THEOREM 1. *Every discrete Möbius group acting on B^n with $V(B^n/G) < \infty$ has qm automorphic mappings.*

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We do not know, whether qm automorphic mappings exist for all discrete Möbius groups in B^n.

4. Let G be a discrete Möbius group acting on B^n with $V(B^n/G)<\infty$, P a simple fundamental polyhedron, $\tilde{P}$ a fundamental set for G with $P\subset\tilde{P}\subset\bar{P}$, $f:B^n\to\bar{R}^n$ an automorphic qm mapping under G and $N(f,\tilde{P})=\sup\operatorname{card} f^{-1}(y)\cap\tilde{P}$ over all $y\in\bar{R}^n$.

THEOREM 2. *Let f, G, P, and $\tilde{P}$ be as above.*

(i) *If U is any open set in R^n which meets ∂B^n, then $\bar{R}^n\setminus f(U\cap B^n)$ is of zero n-capacity.*

(ii) *If $N(f,\tilde{P})<\infty$, then $\bar{R}^n\setminus fB^n$ is of finite cardinality.*

(iii) *If $\bar{P}\subset B^n$ or if $\lim f(x)$, as $x\to p$ in $\tilde{P}$, exists at every boundary vertex $p\in\bar{P}\cap\partial B^n$, then $N(f,\tilde{P})<\infty$ and*

$$\sum_{x\in f^{-1}(y)\cap\tilde{P}} i(x,f)/N(x,G) = N(f,\tilde{P})$$

for all $y\in fB^n$. Here $i(x,f)$ denotes the local topological index of f at x and $N(x,G)=\operatorname{card}\{g\in G:g(x)=x\}$.

(iv) *If $\bar{P}\cap\partial B^n\neq\varnothing$ and $N(f,\tilde{P})<\infty$, then the set Q of all parabolic fixed points of G is dense in ∂B^n and f has a radial limit at every point $b\in Q$.*

THEOREM 3. *Let f, G, P, and $\tilde{P}$ be as above and let $p\in\partial B^n$ be a boundary vertex of P. If $N(f,\tilde{P})<\infty$ and $\lim f(tp)=a\neq\infty$ as $t\to 1$, then for all sufficiently small $r>0$*

$$A_1e^{-\alpha/r}\leqq M(r)\leqq A_2e^{-\beta/r}.$$

Here $M(r)=\sup|f(x)-a|$ over all $x\in B^n$ with $|x-(1-r)p|=r$, α and β are constants which depend on n, G, $N(f,\tilde{P})$, and the dilatations of f, and A_1, A_2 are constants depending on f and G.

The main tools used in the proofs of Theorems 2 and 3 are two capacity inequalities for condensers in B^n/G and general results on open and discrete mappings on manifolds.

5. One of the differences between plane and space qm mappings f is in the structure of their *branch set* B_f (the set of points where f is not a local homeomorphism) (cf. [Z], [**MRV3**, 2.3]). These results combined with information on the geometry of Möbius groups give

THEOREM 4. *Let $f:B^n\to\bar{R}^n$, $n>2$, be a qm automorphic mapping for a Möbius group G acting on B^n with $V(B^n/G)<\infty$. If $\infty\notin fB^n$ or if $N(f,\tilde{P})<\infty$, then $B_f\neq\varnothing$; moreover $\partial B^n\subset\bar{B}_f$.*

The condition $n>2$ is essential (the elliptic modular function is a counterexample), and so is the condition $V(B^n/G)<\infty$. This is shown by an example of a bounded qm local homeomorphism which is automorphic under an infinite Möbius group G with $V(B^n/G)=\infty$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HELSINKI, HELSINKI, FINLAND

DEPARTMENT OF MATHEMATICS, TECHNION, HAIFA, ISRAEL