BULLETIN OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 80, Number 4, July 1974

POSITIVE DEFINITE FUNCTIONS AND VOLTERRA INTEGRAL EQUATIONS[1]

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Communicated by Richard Goldberg, January 3, 1974

1. **Introduction.** The purpose of this research announcement is to describe a new approach for studying asymptotic behavior of solutions of functional equations involving a Volterra operator. More specifically, we study the role played by positive definite and related classes of functions as convolution kernels of the Volterra operators.

2. **Positive and D-positive definite functions.** Let $a(t) \in C(0, \infty) \cap L^1(0, 1)$. We say that $a(t)$ is *positive definite* if for any function $\varphi(t) \in C[0, \infty)$, the quadratic form

$$(1) \qquad Q_a[\varphi](T) = \int_0^T \varphi(t) \int_0^t a(t-\tau)\varphi(\tau)\, d\tau\, dt \geqq 0, \qquad T \geqq 0.$$

Similarly, we say that $a(t)$ is *D-positive definite* if the quadratic form

$$(2) \qquad R_a[\varphi](T) = \int_0^T \varphi(t) \frac{d}{dt} \int_0^t a(t-\tau)\varphi(\tau)\, d\tau\, dt \geqq 0, \qquad T \geqq 0.$$

This definition of positive definite functions differs slightly from that of Bochner since $a(0_+)$ is not assumed to exist and remains finite. The present form, as applied to the study of Volterra integral equations, was first introduced by Halanay [**1**], although he assumed that $a(t) \in C[0, \infty)$, thereby excluding the interesting case $t^{-\nu}$, $0<\nu<1$, the "so-called" Abel kernels. The idea of D-positive definite functions may be found in MacCamy [**6**] although his definition on $a(t)$ is even more restrictive. There is some ambiguity as to what $R_a[\varphi](T)$ means when $a(0_+)$ does not exist. This

AMS (MOS) subject classifications (1970). Primary 45D05, 45M10; Secondary 35B35, 35B40.

Key words and phrases. Positive definite, Fourier transforms, Volterra integral equation, partial differential-functional equation, stability, asymptotic behavior, viscoelasticity.

[1] Research supported in part by Army Research Office, Durham, through Contract No. DA-ARO-D-31-124-72-G95

difficulty is overcome by first restricting R_a to functions in $C^1[0, \infty)$ and then extending to all of $C[0, \infty)$ by passing to the limit.

A special subclass of positive definite functions are the negative exponentials, $\varepsilon e^{-\alpha t}$, $\varepsilon, \alpha > 0$, in terms of which we define a larger subclass called *strongly positive definite* functions. We call the function $a(t)$ strongly positive definite, if there exists $\varepsilon, \alpha > 0$ such that $a(t) - \varepsilon e^{-\alpha t}$ is positive definite. On the other hand, in view of Corollary (b) below, we can consider the special subclass of D-positive definite functions which are nonnegative, nonincreasing, and do not belong to $L^1(0, \infty)$. Denote this special class by $\mathscr{L}$. We define $a(t)$ to be *strongly D-positive definite* if there exists $b(t) \in \mathscr{L}$ such that $a(t) - b(t)$ is D-positive definite.

Criteria for positive and D-positive definiteness of $a(t)$ can be given in terms of its Laplace transform $\hat{a}(s)$ defined by

$$\hat{a}(s) = \lim_{T\to\infty} \int_0^T e^{-st} a(t)\, dt.$$

We now assume in addition that $a(t) \in L^\infty(0, \infty)$ (more precisely, it is sufficient to assume that $a(t)$ belongs to the space of tempered distributions).

THEOREM 1. *Suppose that $\hat{a}(i\omega)$ exists almost everywhere for $\omega \in \mathbf{R}$. Then,*

(a) Re $\hat{a}(i\omega) \geqq 0$ *a.e.* $\Rightarrow a(t)$ *is positive definite, and*

(b) Im $\omega\hat{a}(i\omega) \leqq 0$ *a.e.* $\Rightarrow a(t)$ *is D-positive definite.*

Using Theorem 1, one can now formulate similar criteria for strongly positive and D-positive definite functions. As a consequence of this, we have

COROLLARY. (a) *Let $a(t)$ be nonnegative, $a'(t)$ nonpositive and nondecreasing, then $a(t)$ is positive definite. If, in addition, $a'(t) \not\equiv 0$ then $a(t)$ is strongly positive definite.*

(b) *Let $a(t)$ be nonnegative and nonincreasing, then $a(t)$ is D-positive definite.*

Using these results, one can show that any trigonometric polynomials in cosines with positive coefficients are positive definite, whilst trigonometric polynomials in sines with positive coefficients are D-positive definite. The important class of Abel kernels, $t^{-\nu}$, $0 < \nu < 1$, are both strongly positive and D-positive definite. Furthermore, some weakly singular kernels involving logarithms can also be shown to be strongly positive and strongly D-positive definite. For proofs of these results refer to [7], [8], [9].

3. **Volterra integral equations.** We consider the following two nonlinear Volterra integral equations studied by Levin [**3**], [**4**]:

$$u'(t) = f(t) + \int_0^t a(t-\tau)g(u(\tau))\,d\tau, \tag{3}$$

$$u(t) = f(t) + \int_0^t a(t-\tau)g(u(\tau))\,d\tau, \tag{4}$$

where $f(t) \in L^1(0, \infty)$ and $g(u)$ satisfies

$$g(u) \in C(-\infty, \infty), \qquad ug(u) > 0, \qquad u \neq 0;$$

$$\lim_{|u|\to\infty} G(u) = \infty, \qquad |g(u)| \leqq M(1 + G(u)), \qquad G(u) = \int_0^u g(\xi)\,d\xi.$$

It is known [**2**] that if $a(t)$ is nonnegative, $a'(t)$ nonpositive and nondecreasing, then all solutions of (3) are bounded, and if, in addition, $a'(t) \not\equiv 0$, then all solutions tend to zero. On the other hand, if $a(t)$ is nonnegative, nonincreasing, then all solutions of (4) are bounded, and if, in addition, $a(t) \notin L^1(0, \infty)$, then all solutions tend to zero [**5**]. Using the results given in §1, we can now state (see [**9**])

THEOREM 2. *If $a(t)$ is positive and D-positive definite, respectively, then all solutions of* (3) *and* (4) *are bounded, respectively. Moreover, if $a(t)$ is strongly positive and D-positive definite, respectively, then all solutions of* (3) *and* (4) *tend to zero, respectively.*

4. **An integro-partial differential equation.** The concepts of positive and D-positive definiteness can also be used to study asymptotic behavior of solutions of integro-partial differential equations with equal efficiency. We consider the following initial boundary value problem which arises from the study of viscoelasticity:

$$\begin{aligned} &\frac{\partial u}{\partial t}(x, t) = \int_0^t a(t-\tau)\{\Delta u(x,\tau) + g(\tau, u(x,\tau))\}\,d\tau, \qquad x \in \Omega,\ t \geqq 0, \\ &u(x, t) = 0, \qquad x \in \partial\Omega \quad \text{and} \quad u(x, 0) = u_0(x), \end{aligned} \tag{5}$$

where Ω is a bounded domain in $\boldsymbol{R}^n$ and $g(t, u)$ is a nonlinear perturbing term satisfying

$$|g(t, u)| \leqq \lambda(t)\,|u|^\sigma, \qquad 0 \leqq \sigma \leqq 1,\ \lambda(t) \in L^1(0, \infty).$$

Here we are interested in establishing the existence of solutions of (5) and the asymptotic behavior of these solutions as $t \to \infty$. Denote by $\rho(t)$ the creep compliance function corresponding to $a(t)$, i.e. $\int_0^t \rho(t-\tau)a(t)\,d\tau = t$.

THEOREM 3. *Let $p''(t)$ be positive definite and $\lambda(t) \in L^2(0, T)$, for every finite $T>0$. Then equation* (5) *has a generalized solution $u(x, t)$ in the sense that $u(x, t) \in L^2(0, T; H_0^1(\Omega))$, $\partial u(x, t)/\partial t \in L^2(0, T; L^2(\Omega))$ and satisfies* (5) *weakly in $L^2(\Omega)$. If, in addition, $a(t)$ is strongly positive and $g_u(t, u)$ is bounded, then*

$$\lim_{t\to\infty} \|u(\cdot, t)\|_1 = 0,$$

where $\| \ \|_1$ denotes the norm for $H_0^1(\Omega)$.

The hypotheses required on $a(t)$ and $\rho(t)$ are easily satisfied if $a(t)=\varepsilon e^{-\alpha t}$, $\varepsilon, \alpha>0$. For other examples, we refer the reader to [**10**]. The proof of existence is based upon the Galerkin method and Sobolev's embedding lemma, whereas the asymptotic behavior is derived using Gårding's inequality and a priori estimates for elliptic partial differential operators. Details of these results together with extensions of positive and D-positive definite functions to Hilbert spaces will appear elsewhere [**10**].

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