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HOMOLOGY AND IMAGES OF SEMIANALYTIC SETS

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ABSTRACT. The homology of semianalytic sets may be treated using chains which are themselves locally-finite integral combinations of disjoint, oriented semianalytic submanifolds. The analytic image of a relatively compact semianalytic set, though not necessarily semianalytic, admits a finite stratification into connected analytic submanifolds of various dimensions.

A subset A of a (real) analytic manifold M is called *analytic* (respectively, *semianalytic*) if M can be covered by open sets U for which there is a real-valued function f (respectively, a finite family $\mathscr{F}$ of real-valued functions) analytic in U so that $U \cap A$ equals $f^{-1}\{0\}$ (respectively, $U \cap A$ is a union of connected components of $f^{-1}\{0\} \sim g^{-1}\{0\}$ for some $f, g \in \mathscr{F}$). A *stratum* in M is a connected (properly embedded) differentiable submanifold of M. A *stratification* $\mathscr{S}$ of a subset A of M is a locally finite partition of A into strata $\mathscr{S}$ so that $(A \cap \operatorname{Clos} S) \sim S$ is a union of strata in $\mathscr{S}$ having dimension less than the dimension of S. It is well known [**9**, §13], [**7**, 2.8] that every semianalytic set admits a stratification into semianalytic strata.

A j-dimensional *analytic chain* T in M is a sum of integral multiples of oriented j-dimensional semianalytic strata belonging to some fixed stratification of M. Since the restriction to these strata of j-dimensional Hausdorff measure is locally-finite by [**2**, 3.4.8(13)], the analytic chain T is (by oriented integration, counting multiplicities, of differential j forms of compact support in M) a j-dimensional current in M. The set spt T, being the union of the closures of the strata occurring with nonzero multiplicity, is semianalytic. For $j \geqq 1$, the $(j-1)$-dimensional current ∂T, defined by $\partial T(\psi) = T(d\psi)$ for $\psi \in \mathscr{D}^{j-1}(M)$, is, by [**2**, 4.2.28], also an analytic chain in M.

Suppose $M \supset A \supset B$. Using the group of *real analytic cycles* $\mathscr{Z}_j(A, B) = \{T : T$ is a j-dimensional analytic chain of compact support, spt $T \subset A$,

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and spt $\partial T \supset B$ or $j=0\}$, the subgroup of *real analytic boundaries*, $\mathscr{B}_j(A, B)=\{R+\partial S: R \in \mathscr{Z}_j(B, B)$ and $S \in \mathscr{Z}_{j+1}(A, A)\}$, and the *real analytic homology group*, $\boldsymbol{H}_j(A, B)=\mathscr{Z}_j(A, B)/\mathscr{B}_j(A, B)$, we obtain in [**6**, 4.6–4.7] the following

THEOREM 1. *If $A \supset B$ are semianalytic sets, then there exists an arbitrarily small open neighborhood W of B such that $\boldsymbol{H}_j(A \cap W, B) \simeq 0$ for all j.*

COROLLARY 1. *There exist arbitrarily small open neighborhoods U of A in M and V of B in U so that the inclusion of $\mathscr{Z}_j(A, B)$ into $\mathscr{Z}_j(U, V)$ induces an isomorphism, $\boldsymbol{H}_j(A, B) \simeq \boldsymbol{H}_j(U, V)$, for all j.*

This allows us to define in [**6**, §5], by approximation, the homomorphism $\boldsymbol{H}_j(f): \boldsymbol{H}_j(C, D) \to \boldsymbol{H}_j(A, B)$ for any continuous map $f: (C, D) \to (A, B)$ where $C \supset D$ are semianalytic subsets of an analytic manifold; the axioms of Eilenberg-Steenrod follow as in [**2**, 4.4.1]. Also in [**6**, §6] a homology intersection product, for oriented M,

$$\bigcap : \boldsymbol{H}_i(A, B) \times \boldsymbol{H}_j(A, B) \to \boldsymbol{H}_{i+j-\dim M}(A, B)$$

where $i+j \geqq \dim M$, results by use of the intersection theory for real analytic chains of [**4**, §5]. The proofs of [**6**, §§2–6] all carry over for an analogous treatment of the homology of semialgebraic sets by real algebraic chains or for the homology with $\boldsymbol{Z}/\nu\boldsymbol{Z}$ coefficients, where $\nu \in \{2, 3 \cdots\}$, of semianalytic sets by real analytic chains modulo ν [**5**].

A. Borel and A. Haefliger, employing the Borel-Moore homology for locally-compact spaces proved the orientability modulo 2 of real analytic sets and established a formula equating the modulo 2 cycle of the real part of the intersection of two holomorphic varieties with the intersection of the modulo 2 cycles of the real parts of the varieties. These facts are reproven in [**6**, §§6, 8] using analytic chains and Federer's theory of slicing [**2**, 4.3], [**4**, §4]. The proof of Theorem 1 involves, for bounded semianalytic subsets of $\boldsymbol{R}^n$, a certain stratification and system of neighborhoods built up from finitely many local normal decompositions of Łojasiewicz [**9**, §13]. We do not make use of the triangulability of semianalytic sets which is established in [**3**] and [**8**]. Other interesting aspects of semianalytic sets are treated in [**1**], [**2**, 3.4.5–3.4.11], [**9**], [**10**], [**11**, §4] and [**12**], [**9**] being the most informative.

Even though semianalytic sets are closed under finite union, intersection, complement, cartesian product, and inverse image under analytic maps, the analytic image of even a compact analytic manifold may fail to be semianalytic [**9**, p. 135]. A subset C of M is called a *semianalytic shadow* if M can be covered by open sets U for which there is an analytic manifold P, an analytic mapping $p: P \to M$, and a finite family $\mathscr{A}$ of relatively compact semianalytic subsets of P so that $U \cap C$ is a union of connected components of $p(A) \sim p(B)$ for some $A, B \in \mathscr{A}$.

THEOREM 2. *For any locally finite family $\mathscr{C}$ of semianalytic shadows in M, there is a stratification $\mathscr{S}$ of M into semianalytic shadows so that each member of $\mathscr{C}$ is a union of strata in $\mathscr{S}$.*

THEOREM 3. *If $f: M \to N$ is a proper analytic mapping, then $\mathscr{S}$ may be chosen so that $\{f(S): S \in \mathscr{S}\}$ extends to a stratification of N and $f|S$ is one-one for all $S \in \mathscr{S}$ with* $\dim f(S) = \dim S$.

The proofs of these statements in [7] involve certain semianalytic stratifications, the rank theorem, a cartesian product construction of [**4**, 2.8], and induction on the Hausdorff dimension of $\bigcup \mathscr{C}$. The stratification $\mathscr{S}$ may also be refined to satisfy Whitney condition (B) [**11**, §3]. Statements similar to Theorem 2 and Corollary 2 are given without formal proof in [**12**, III B–C].

We have recently learned of the interesting work of H. Hironaka [**13**] and [**14**] on semianalytic shadows (which he calls subanalytic sets). Using his theory of resolution of singularities and blowing up techniques, he has [**13**] established Theorem 2. Because of differences in proofs as well as the discussions of [7, §2, §5], our article [7] may be of independent interest.

REFERENCES

1. D. Burghelea and A. Verona, *Local homological properties of analytic sets*, Manuscripta Math. **7** (1972), 55–66.

2. H. Federer, *Geometric measure theory*, Die Grundlehren der math. Wissenschaften, Band 153, Springer-Verlag, Berlin and New York, 1969. MR **41** #1976.

3. B. Giesecke, *Simpliziale Zerlegung abzählbarer analytischer Räume*, Math. Z. **83** (1964), 177–213. MR **28** #2563.

4. R. Hardt, *Slicing and intersection theory for chains associated with real analytic varieties*, Acta Math. **129** (1972), 75–136.

5. ———, *Slicing and intersection theory for chains modulo v associated with real analytic varieties*, Trans. Amer. Math. Soc. **183** (1973), 327–340.

6. ———, *Homology theory for real analytic and semianalytic sets*, Ann. Scuola Norm. Sup. Pisa (to appear).

7. ———, *Stratification of real analytic mappings and images*, (preprint).

8. S. Łojasiewicz, *Triangulation of semi-analytic sets*, Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 449–474. MR **30** #3478.

9. ———, *Ensembles semianalytiques*, Cours Faculté des Sciences d'Orsay, Inst. Haute Etudes Sci., Bures-sur-Yvette, 1965.

10. ———, *Une propiété topologique des sous-ensembles analytique reéls*, Colloques internationaux du Centre National de la Recherche Scientifique, No. 117, Les Equations aux Dérivées Partielles, Paris, 1972, pp. 87–89.

11. J. Mather, *Stratifications and mappings*, Harvard University notes, Cambridge, Mass., 1971.

12. R. Thom, *Ensembles et morphismes stratifiés*, Bull. Amer. Math. Soc. **75** (1969), 240–284. MR **39** #970.

13. H. Hironaka, *Subanalytic sets*, Number Theory, Algebraic Geometry, and Commutative Algebra (Dedicated to Akizuki), Kinokunia, Tokyo, Japan, 1973, pp. 453–493.

14. ———, *Subanalytic sets*, Lecture notes of Istituto matematico, Leonida Tonelli, Pisa, 1973.

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